Your Turn

We want to use the following distribution for inference, where we know the shape, but not the full distributional form.

$\tilde{f}(x) = c \cdot \frac{\log(x)}{1+x^2} \underbrace{\mathbb{I}(x \in [1,\pi])}$

$$\tilde{f}(x) = c\frac{\log(x)}{1+x^2}, \quad x \in [1, \pi]$$

What do we need for this to be a valid pdf?

We need $\int_1^{\pi} c \frac{\log(x)}{1+x^2} dx = 1.$

$\Rightarrow \int_1^{\pi} \frac{\log x}{1+x^2} dx = \frac{1}{c}$

1. estimate c using MC integration with $f \sim Unif(1,\pi)$, $m = 10000$.

a) Find f and g (write integral as expectation of $g(X)$, $X \sim f$).

$f(x) = \begin{cases} \frac{1}{\pi - 1} & x \in [1,\pi] \\ 0 & \text{o.w.} \end{cases}$

need to squeeze f in here.

$\int_1^{\pi} \frac{\log x}{1+x^2} dx = \int_1^{\pi} \underbrace{\frac{\log x}{1+x^2} \frac{1}{\frac{1}{\pi-1}}}_{g} \frac{1}{\pi - 1} dx = E\left[\frac{\log X}{1+X^2} \cdot (\pi - 1)\right]$, $X \sim Unif(1, \pi)$. ✓

b) Make plan!

1. Sample $X_1, \dots, X_m$ from $Unif(1,\pi)$.
2. $\hat{\theta} = \frac{1}{m}\sum_{i=1}^{m} g(X_i)$. where $g(x) = \frac{\log x}{1+x^2}(\pi - 1)$.

c) do it!

↳ easy to also estimate $Var(\hat{\theta})$.

2. Can we do better? Importance sampling w/ $\phi \sim N(1,1)$.

Make a plan ⟶ think about is this a good idea?

plot $\frac{f(x)g(x)}{f(x)}$, $\frac{f(x)g(x)}{\phi(x)}$

want most "constant" function.

1. Sample $Y_1, \dots, Y_m \sim N(1,1)$.
2. $\hat{\theta}' = \frac{1}{m}\sum_{i=1}^{m} g(Y_i)\frac{f(Y_i)}{\phi(Y_i)}$

$\int_1^{\pi} \frac{\log(x)}{1+x^2} \cdot (\pi - 1) \cdot \frac{1}{\pi - 1} \cdot \frac{\phi(x)}{\phi(x)} dx$

$\int_{-\infty}^{\infty} \frac{\log(x)}{1+x^2}$

③ $\tilde{f}(x) = \dfrac{\dfrac{\log(x)}{1+x^2}}{\displaystyle\int_1^{\pi} \frac{\log(x)}{1+x^2}\,dx}$ for $x \in [1, \pi]$.

have $\hat{\theta}$

How to sample from $\tilde{f}$?

Accept - reject!

Need to choose $e(x)$ envelope!

Recall $e(x) = d \cdot \tilde{g}(x)$ let $\tilde{g} \sim \text{Unif}[1, \pi]$.

$\Rightarrow \tilde{g}(x) = \frac{1}{\pi - 1}, x \in [1, \pi]$.

need $e(x) \geq f(x)$

find d based on max value of $\tilde{f}(x)$ between $[1, \pi]$.

1. Find $e(x)$ [written as a function of $\max \tilde{f}(x)$].
2. get $n=1000$ samples from $\tilde{f}(x)$ using accept-reject.

need $e(x) \geq \tilde{f}(x) \ \forall x$

$\Rightarrow e(x) = \max \tilde{f}(x)$ will work, i.e. $d \cdot \frac{1}{\pi - 1} = \max \tilde{f}(x)$

$d = (\pi - 1) \max \tilde{f}(x)$.