2 Monte Carlo Methods for Hypothesis Tests

There are two aspects of hypothesis tests that we will investigate through the use of Monte Carlo methods: Type I error and Power.

Example 2.1 Assume we want to test the following hypotheses

$$H_0 : \mu = 5$$
$$H_a : \mu > 5$$

with the test statistic

$$T^* = \frac{\overline{x} - 5}{s/\sqrt{n}}.$$

This leads to the following decision rule:

critical value (quantile).

Reject H_0 if $T^* > t_{(1-\alpha/2), n-1} = qt(1-\alpha/2, n-1)$.

equivalent to: Reject H_0 if p-value $< \alpha$.

What are we assuming about X?

$X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ unknown.

(with big enough n & with help from CLT $X_1, \dots, X_n \overset{iid}{\sim} F_X$, unknown but finite variance).

2.1 Types of Errors

Type I error: Reject H_0 when H_0 true.

Type II error: Fail to reject H_0 when H_0 false.

TRUTH

Decision	H_0 true	H_0 false.
Reject H_0	Type I error α	Correct decision power $= 1 - \beta$
Fail to Reject H_0	Correct decision.	Type II error β

$\alpha = P(\text{type I error})$
$= P(\text{reject } H_0 \mid H_0 \text{ true})$

$\beta = P(\text{type II error})$
$= P(\text{Fail to reject } H_0 \mid H_0 \text{ false}).$

Usually we set $\alpha = 0.05$ or 0.10, and choose a sample size large enough. such that power $= 1 - \beta \geq 0.80$.

For simple cases, we can find formulas for α and β.

For all others, we can use Monte carlo integration to estimate $\underset{\text{type I error rate}}{\alpha}$ & $\underset{\text{power.}}{1-\beta}$

2.2 MC Estimator of α

Assume $X_1, \dots, X_n \sim F(\theta_0)$ (i.e., assume H_0 is true).

Then, we have the following hypothesis test –

$$H_0 : \theta = \theta_0$$
$$H_a : \theta > \theta_0$$

and the statistics T^*, which is a test statistic computed from data. Then we **reject** H_0 if $T^* >$ the critical value from the distribution of the test statistic.

This leads to the following algorithm to estimate the Type I error of the test (α)

For replicate $j = 1, \dots, m$

1. Generate $X_1^{(j)}, \dots, X_n^{(j)} \sim F(\theta_0)$
2. Compute $T^{*(j)} = \Psi(X_1^{(j)}, \dots, X_n^{(j)})$ a function of the data
3. Let $I_j = \begin{cases} 1 & \text{if reject } H_0 \text{ based on } T^{*(j)} \\ 0 & \text{o.w.} \end{cases}$

Then $\hat{\alpha} = \frac{1}{m} \sum_{j=1}^{m} I_j$ = estimated Type I error $= \hat{P}(\text{reject } H_0 \mid H_0 \text{ true})$

For iid draws from $F(\theta_0)$

and $se(\hat{\alpha}) = \sqrt{\frac{\hat{\alpha}(1-\hat{\alpha})}{m}}$ = estimate of $\sqrt{Var(\hat{\alpha})}$ = estimated uncertainty in estimate of α.

Why? $Var(\hat{\alpha}) = Var\left(\frac{1}{m}\sum_{j=1}^{m} I_j\right) = \frac{1}{m^2}\sum_{j=1}^{m} Var\, I_j$ and $I_j \sim$ Bernoulli (p) where

$p = P(\text{reject } H_0 \text{ based on } T^*)$
$= P(\text{reject} \mid X_1, \dots, X_n \sim F(\theta_0)) = \alpha.$

$\Rightarrow Var(I_j) = \alpha(1-\alpha)$

and $Var(\hat{\alpha}) = \frac{1}{m}\alpha(1-\alpha) \Rightarrow \widehat{Var}(\hat{\alpha}) = \frac{1}{m}\hat{\alpha}(1-\hat{\alpha}).$

Your Turn

Example 2.2 (Pearson's moment coefficient of skewness) Let $X \sim F$ where $E(X) = \mu$ and $Var(X) = \sigma^2$. Let

$$\sqrt{\beta_1} = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right].$$

Then for a

- symmetric distribution, $\sqrt{\beta_1} = 0$,
- positively skewed distribution, $\sqrt{\beta_1} > 0$, and
- negatively skewed distribution, $\sqrt{\beta_1} < 0$.

The following is an estimator for skewness

$$\sqrt{b_1} = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^3}{\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2\right]^{3/2}}.$$

It can be shown by Statistical theory that if $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, then as $n \to \infty$,

$$\sqrt{b_1} \dot{\sim} N\left(0, \frac{6}{n}\right). \quad \Rightarrow \quad \frac{\sqrt{b_1}}{\sqrt{\frac{6}{n}}} \dot{\sim} N(0,1).$$

Thus we can test the following hypothesis

$$\begin{cases} H_0 : \sqrt{\beta_1} = 0 \\ H_a : \sqrt{\beta_1} \neq 0 \end{cases}$$

H_0: symmetric distribution
H_a: not symmetric.

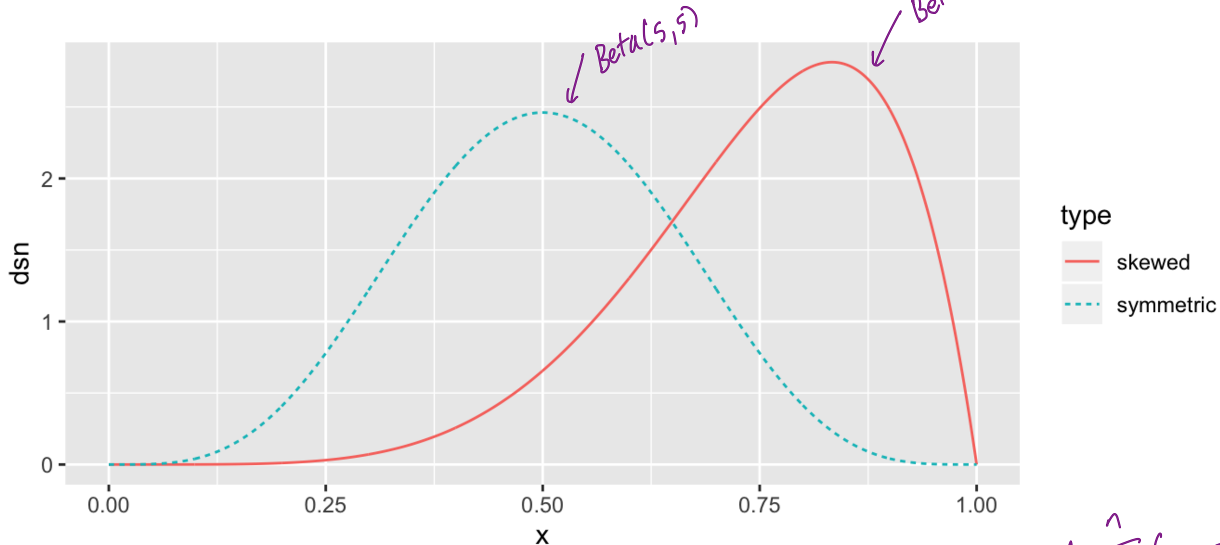
by comparing $\frac{\sqrt{b_1}}{\sqrt{\frac{6}{n}}}$ to a critical value from a $N(0,1)$ distribution.

In practice, convergence of $\overset{dsn}{\sqrt{b_1}}$ to a $N\left(0, \frac{6}{n}\right)$ is slow.

$\Rightarrow$ n needs to be very large for dist of $\sqrt{b_1}$ $\overset{approx}{\sim}$ Normal.

We want to assess $P(\text{Type I error})$ for $\alpha = 0.05$ for $n = 10, 20, 30, 50, 100, 500$.

```r
library(tidyverse)

# compare a symmetric and skewed distribution
data.frame(x = seq(0, 1, length.out = 1000)) %>%
  mutate(skewed = dbeta(x, 6, 2),
         symmetric = dbeta(x, 5, 5)) %>%
  gather(type, dsn, -x) %>%
  ggplot() +
  geom_line(aes(x, dsn, colour = type, lty = type))
```



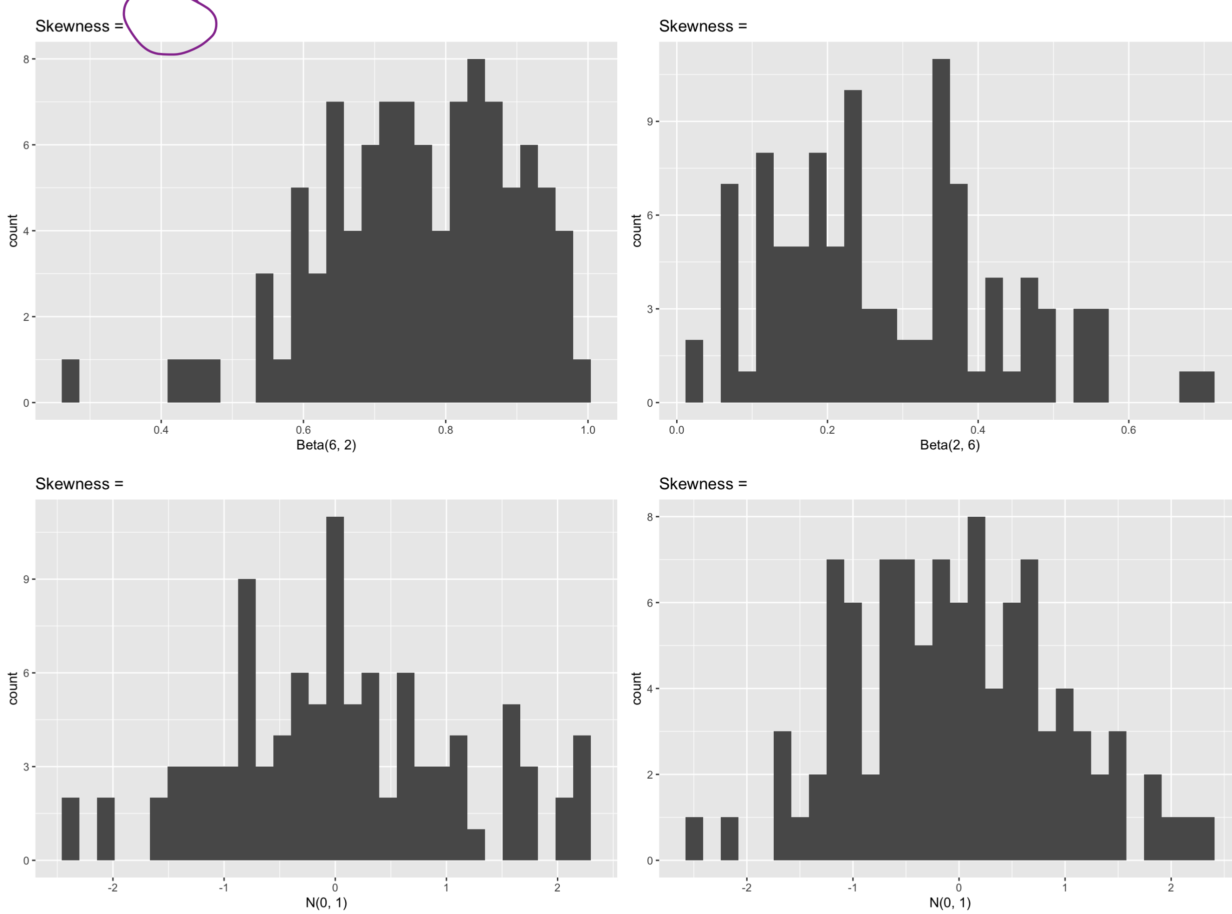
$$\boxed{\sqrt{b_1}} = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^3}{\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right]^{3/2}}$$

```r
## write a skewness function based on a sample x
skew <- function(x) {
  YOUR TURN
}

## check skewness of some samples
n <- 100
a1 <- rbeta(n, 6, 2)
a2 <- rbeta(n, 2, 6)

## two symmetric samples
b1 <- rnorm(100)
b2 <- rnorm(100)

## fill in the skewness values
ggplot() + geom_histogram(aes(a1)) + xlab("Beta(6, 2)") +
  ggtitle(paste("Skewness = "))
```

add in skewness value for the samples

```r
ggplot() + geom_histogram(aes(a2)) + xlab("Beta(2, 6)") +
  ggtitle(paste("Skewness = "))
ggplot() + geom_histogram(aes(b1)) + xlab("N(0, 1)") +
  ggtitle(paste("Skewness = "))
ggplot() + geom_histogram(aes(b2)) + xlab("N(0, 1)") +
  ggtitle(paste("Skewness = "))
```

add stat value.




```r
## Assess the P(Type I Error) for alpha = .05, n = 10, 20, 30, 50,
  100, 500
```

YOUR TURN

Example 2.3 (Pearson's moment coefficient of skewness with variance correction) One way to improve performance of this statistic is to adjust the variance for small samples. It can be shown that

$$Var(\sqrt{b_1}) = \frac{6(n-2)}{(n+1)(n+3)}.$$

Assess the Type I error rate of a skewness test using the finite sample correction variance.

YOUR TURN