Chapter 2: Probability for Statistical Computing

Just like we did for R.

We will **briefly** review some definitions and concepts in probability and statistics that will be helpful for the remainder of the class.

Just like we reviewed computational tools (`R` and packages), we will now do the same for probability and statistics.

Note: This is not meant to be comprehensive. I am assuming you already know this and maybe have forgotten a few things. i.e. you may need more refreshing outside of class...

https://xkcd.com/892/

Alternative text: "Hell, my eighth grade science class managed to conclusively reject it just based on a classroom experiment. It's pretty sad to hear about million-dollar research teams who can't even manage that."

1 Random Variables and Probability

Definition 1.1 A *random variable* is a function that maps sets of all possible outcomes of an experiment (sample space Ω) to $\mathbb{R}$.

Example 1.1

Toss 2 dice

X = sum of the values on top of dice.

r.v.

Example 1.2

Randomly select 25 deer and test for CWD (chronic wasting disease)

Ω = Sample space = $\{+, -\ \text{CWD}\}$

r.v. $X_i = \begin{cases} 1 & \text{if test is } + \\ 0 & \text{if test is } - \end{cases}$ observe $X_1, \ldots, X_{25}$

$P = \sum_{i=1}^{25} X_i / 25$ is also a RV!

Example 1.3

Today's high temperature $= X_i$

Types of random variables –

Discrete take values in a countable set.

Ex 1.1 and X_i from Ex 1.2

Continuous take values in an uncountable set (like $\mathbb{R}$)

real numbers $(-\infty, \infty)$.

Ex. 1.3, $x_i \in \mathbb{R}$

P from Ex 1.2, $P \in [0,1]$.

1.1 Distribution and Density Functions

Definition 1.2 The *probability mass function (pmf)* of a discrete random variable X is f_X defined by

sometimes when r.v. is obvious we will omit the subscript.

$$f_X(x) = P(X = x)$$

where $P(\cdot)$ denotes the probability of its argument.

There are a few requirements of a **valid** pmf

1. $f(x) \geq 0 \quad \forall x \in \mathcal{X}.$
2. $\sum_{\mathcal{X}} f(x) = 1$
3. We call $\mathcal{X} = \{x : f(x) > 0\}$ the "support" of X.

not a requirement

Example 1.4 Let Ω = all possible values of a roll of a single die $= \{1, \ldots, 6\}$ and X be the outcome of a single roll of one die $\in \{1, \ldots, 6\}$.

$f(1) = P(X=1) = \frac{1}{6}$
$\vdots$
$f(6) = 1/6$

① ✓ $f(x) \geq 0 \quad \forall x \in \mathcal{X}$

② ✓ $\sum_{\mathcal{X}} f(x) = 1$

A pmf is defined for **discrete variables**, but what about **continuous**? Continuous variables do not have positive probability ~~pass~~ mass at any single point.

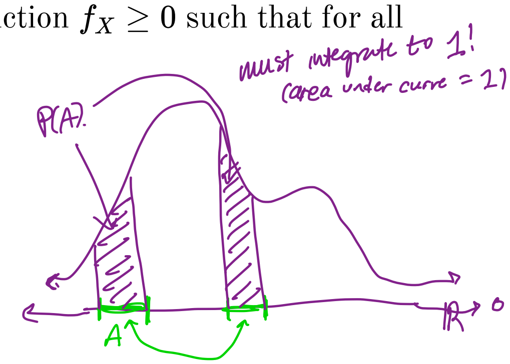
Definition 1.3 The *probability density function (pdf)* of a random variable X is f_X defined by

$$P(X \in A) = \int_{x \in A} f_X(x) dx. \qquad \text{where } A \subset \mathbb{R}$$

X is a continuous random variable if there exists this function $f_X \geq 0$ such that for all $x \in \mathbb{R}$, this probability exists.

For f_X to be a valid pdf,

1. $f(x) \geq 0$ for all x
2. $\int_{\mathbb{R}} f(x) dx = 1.$



Again $\mathcal{X} = \{x : f(x) > 0\}$ is the "support" of X.

There are many named pdfs and cdfs that you have seen in other class, e.g.

Binomial, geometric, bernoulli, Poisson, Normal, Beta, Gamma, exponential

Example 1.5 Let

$\mathcal{X}$ support

$$f(x) = \begin{cases} c(4x - 2x^2) & \boxed{0 < x < 2} \\ 0 & \text{otherwise} \end{cases}$$

s.t. $f(x)$ is a valid pdf.

"normalizing constant"

Find c and then find $P(X > 1)$

$1 = \int_{\mathbb{R}} f(x)\,dx = \int_0^2 c(4x-2x^2)\,dx = c\left[2x^2 - \frac{2x^3}{3}\right]_0^2 = c\left[\frac{8}{3}\right] \Rightarrow c = \frac{3}{8}$

$P(X>1) = \int_1^\infty f(x)\,dx = \int_1^2 \frac{3}{8}(4x-2x^2)\,dx = \frac{3}{8}\left[2x^2 - \frac{2x^3}{3}\right]_1^2 = \frac{1}{2}$

both cts and discrete.

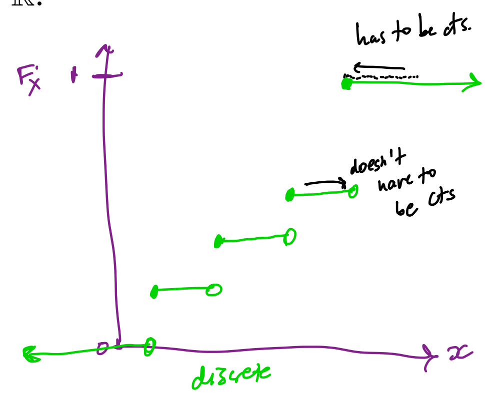
Definition 1.4 The *cumulative distribution function (cdf)* for a random variable X is F_X defined by

$$F_X(x) = P(X \leq x), \quad x \in \mathbb{R}.$$

r.v. r.v.

The valid cdf has the following properties

1. F_X is non-decreasing.
2. F_X is right continuous.
3. $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$.



A random variable X is *continuous* if F_X is a continuous function and *discrete* if F_X is a step function.

Example 1.6 Find the cdf for the previous example.

$F_X(x) = P(X \leq x) \quad x \in \mathbb{R}.$

what if $x \leq 0$? $F_X(x) = 0$

what if $x \geq 2$? $F_X(x) = 1$

for $x \in (0,2)$, $P(X \leq x) = \int_0^x \frac{3}{8}(4y - 2y^2)\,dy = \frac{3}{8}\left[2y^2 - \frac{2y^3}{3}\right]_0^x = \frac{3}{4}x^2\left(1 - \frac{x}{3}\right)$

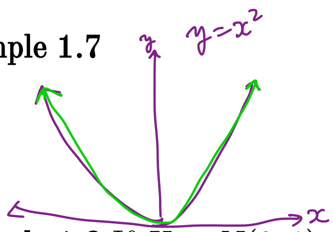
Note $f(x) = F'(x) = \frac{dF(x)}{dx}$ in the continuous case.

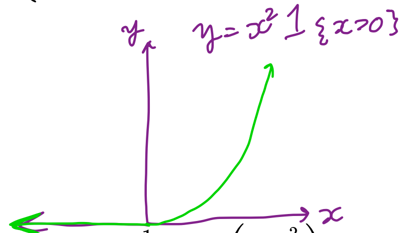
$\Rightarrow F_X(x) = \begin{cases} 0 & x \leq 0 \\ 3/4\, x^2(1-\frac{x}{3}) & x \in (0,2) \\ 1 & x \geq 2. \end{cases}$

Recall an indicator function is defined as

$$1_{\{A\}} = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{otherwise} \end{cases}.$$

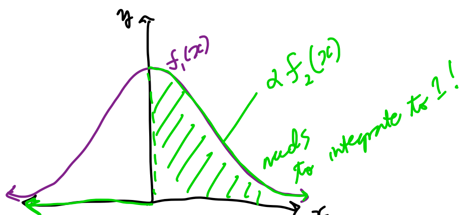
Example 1.7





Example 1.8 If $X \sim N(0, 1)$, the pdf is $f_1(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ for $-\infty < x < \infty$.

If $f_2(x) = \frac{c}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) 1_{\{x>0\}}$, what is c? for $f_2(x)$ to be a valid pdf?



Need $1 = \int_0^\infty \frac{c}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$.

We know symmetry!

$\Rightarrow \int_0^\infty f_1(x)dx = \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = \frac{1}{2}$.

$1 = \int_0^\infty \frac{c}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = \frac{c}{2}$.

$\Rightarrow$ need $c = 2$.

1.2 Two Continuous Random Variables

Definition 1.5 The *joint pdf* of the continuous vector (X, Y) is defined as

$$P((X, Y) \in A) = \iint_A f_{X,Y}(x, y)dxdy$$

for any set $A \subset \mathbb{R}^2$.

Joint pdfs have the following properties

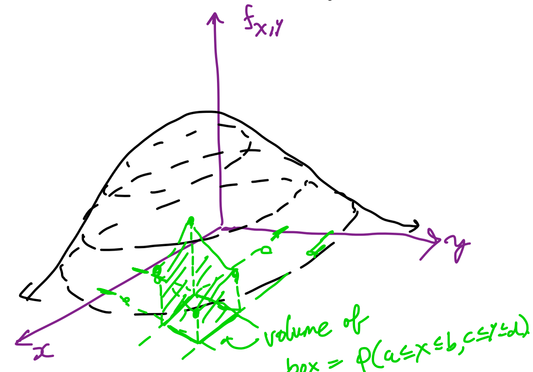
1. $f_{X,Y}(x,y) \geq 0 \quad \forall x, y$.
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$

Note we can also have joint pmf's for discrete variables.
$\sum\sum_{\mathcal{X}} f(x,y) = 1$.

and a support defined to be $\{(x, y) : f_{X,Y}(x, y) > 0\} = \mathcal{X}$.

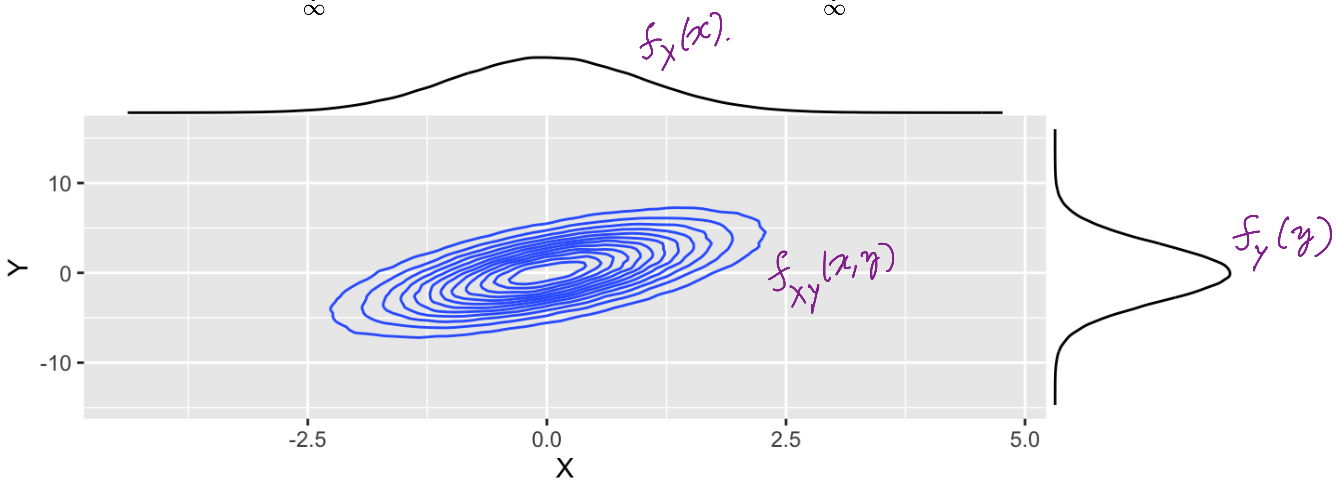
Example 1.9

$$P(a \le X \le b,\ c \le Y \le d) = \int_a^b \int_c^d f_{XY}(x,y)\,dy\,dx.$$



The *marginal densities* of X and Y are given by

$$f_X(x) = \int_{\infty}^{\infty} f_{X,Y}(x,y)dy \quad \text{and} \quad f_Y(y) = \int_{\infty}^{\infty} f_{X,Y}(x,y)dx;$$



Example 1.10 (From Devore (2008) Example 5.3, pg. 187) A bank operates both a drive-up facility and a walk-up window. On a randomly selected day, let X be the proportion of time that the drive-up facility is in use and Y is the proportion of time that the walk-up window is in use.

The the set of possible values for (X, Y) is the square $D = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$. Suppose the joint pdf is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & x \in [0,1], y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Evaluate the probability that both the drive-up and the walk-up windows are used a quarter of the time or less.

$$P(\underbrace{\text{drive up used}}_{X} \le \tfrac{1}{4} \text{ and } \underbrace{\text{walk up used}}_{Y} \le \tfrac{1}{4}) = P(X \le \tfrac{1}{4}, Y \le \tfrac{1}{4})$$

$$= \int_0^{1/4}\int_0^{1/4} \frac{6}{5}(x+y^2)\,dx\,dy$$

$$= \int_0^{1/4} \frac{6}{5}\left[\frac{x^2}{2} + xy^2\right]_{x=0}^{x=1/4} dy$$

$$= \int_0^{1/4} \frac{6}{5}\left[\frac{1}{32} + \frac{y^2}{4}\right] dy$$

$$= \frac{6}{5}\left[\frac{y}{32} + \frac{y^3}{12}\right]_0^{1/4} = \frac{6}{5}\left[\frac{1}{32}\cdot\frac{1}{4} + \frac{1}{12}\cdot\left(\frac{1}{4}\right)^3\right] = \frac{7}{640} = 0.0109.$$

Find the marginal densities for X and Y.

$$f_X(x) = \int_0^1 \frac{6}{5}(x+y^2)\,dy = \frac{6}{5}\left[xy + \frac{y^3}{3}\right]_{y=0}^{1} = \begin{cases} \frac{6}{5}\left(x+\frac{1}{3}\right) & \text{for } x\in[0,1] \\ 0 & \text{o.w.} \end{cases}$$

$$f_Y(y) = \int_0^1 \frac{6}{5}(x+y^2)\,dx = \frac{6}{5}\left[\frac{x^2}{2} + xy^2\right]_{x=0}^{1} = \begin{cases} \frac{6}{5}\left(\frac{1}{2}+y^2\right) & \text{for } y\in[0,1] \\ 0 & \text{o.w.} \end{cases}$$

X

Compute the probability that the drive-up facility is used a quarter of the time or less.

$$P\left(X \le \frac{1}{4}\right) = \int_0^{1/4} f_X(x)\,dx = \int_0^{1/4} \frac{6}{5}\left(x+\frac{1}{3}\right)dx$$

$$= \frac{6}{5}\left[\frac{x^2}{2} + \frac{x}{3}\right]_0^{1/4}$$

$$= \frac{6}{5}\left(\left(\frac{1}{4}\right)^2\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{3}\right) = \frac{11}{80} = 0.1375.$$

2 Expected Value and Variance

Definition 2.1 The *expected value* (average or mean) of a random variable X with pdf or pmf f_X is defined as

$$E[X] = \begin{cases} \sum_{x \in \mathcal{X}} x f_X(x_i) & X \text{ is discrete} \\ \int_{x \in \mathcal{X}} x f_X(x) dx & X \text{ is continuous.} \end{cases}$$

Where $\mathcal{X} = \{x : f_X(x) > 0\}$ is the support of X.

This is a weighted average of all possible values $\mathcal{X}$ by the probability distribution.

Example 2.1 Let $\overset{\text{r.v.}}{X} \sim \text{Bernoulli}(p)$. Find $E[X]$. parameter

$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{o.w.} \end{cases} \Rightarrow f(x) = \begin{cases} p & \text{when } x = 1 \\ 1-p & \text{when } x = 0 \end{cases}$ or $f(x) = p^x (1-p)^{1-x}$ pmf $\quad x \in \{0, 1\}$ support $\mathcal{X}$

$$E[X] = \sum_{x \in \mathcal{X}} x f_X(x) = 0 \cdot (1-p) + 1 \cdot (p) = p.$$

$\int u\,dv = uv - \int v\,du$

pick u and dv.

Example 2.2 Let $X \sim \text{Exp}(\lambda)$. Find $E[X]$.

Need integration by parts!! (HW 3).

$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{o.w.} \end{cases}$

$$E[X] = \int_{\mathcal{X}} \underset{u}{x} f(x)\, dx = \int_0^\infty \underset{u}{x} \cdot \underbrace{\boxed{\lambda e^{-\lambda x}\, dx}}_{dv}$$

Definition 2.2 Let $g(X)$ be a function of a continuous random variable X with pdf f_X. Then,

$$E[g(X)] = \int_{x \in \mathcal{X}} g(x) f_X(x) dx.$$

sometimes this is hard (impossible) to compute by hand!

$\Rightarrow$ We will need computing to help estimate this value. (Ch. 5).

corresponding discrete definition X has pmf $f_X(x)$, $E(g(X)) = \sum_{\mathcal{X}} g(x) f_X(x)$.

Definition 2.3 The *variance* (a measure of spread) is defined as

$g(X) = (X - E[X])^2$

$$\begin{aligned} Var[X] &= E\left[(X - E[X])^2\right] \\ &= E[X^2] - (E[X])^2 \end{aligned}$$

simplified

$\star$ computationally friendly formula.

Example 2.3 Let X be the number of cylinders in a car engine. The following is the pmf function for the size of car engines.

x	4.0	6.0	8.0
f	0.5	0.3	0.2

Who might care about this?

car parts manufacturer

car parts distributor

EPA?

Find

$E[X] = \sum_{x} x f(x) = 4(0.5) + 6(0.3) + 8(0.2) = 5.4$

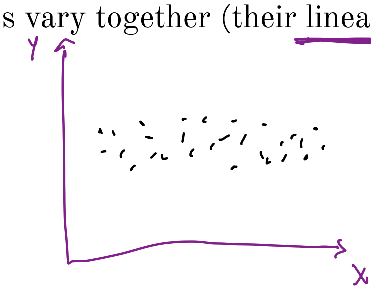
$Var[X] = E[X^2] - [EX]^2$

$E[X^2] = \sum_{x} x^2 f(x) = 4^2(0.5) + 6^2(0.3) + 8^2(0.2) = 31.6$

$\Rightarrow Var(X) = 31.6 - (5.4)^2 = 2.44$ easier to interpret: $sd(X) = \sqrt{Var(X)} = 1.56$

Covariance measures how two random variables vary together (their linear relationship).

$cov[X,Y] > 0$



$cov[X,Y] \approx 0$.

"random noise"

Definition 2.4 The *covariance* of X and Y is defined by

$$\begin{aligned} Cov[X,Y] &= E\left[(X - E[X])(Y - E[Y])\right] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

Note, for 2 cts. r.v.'s,

$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)\,dx\,dy$

2 discrete rvs:

$E[g(X,Y)] = \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} g(x,y) f_{XY}(x,y)$

and the *correlation* of X and Y is defined as

$$\rho(X,Y) = \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}}. \in [-1,1]$$

Two variables X and Y are *uncorrelated* if $\rho(X,Y) = 0$.

no linear relationship

3 Independence and Conditional Probability

In classical probability, the *conditional probability* of an event A given that event B has occured is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}. \quad \longrightarrow \quad P(A|B)P(B) = P(A \cap B).$$

Definition 3.1 Two events A and B are *independent* if $P(A|B) = P(A)$. The converse is also true, so

$$A \text{ and } B \text{ are independent} \Leftrightarrow P(A|B) = P(A) \Leftrightarrow P(A \cap B) = P(A) \; P(B)$$

Theorem 3.1 (Bayes' Theorem) Let A and B be events. Then,

$$P(A|B) \overset{\text{def'n of cond. prb}}{=} \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B)}.$$

3.1 Random variables

The same ideas hold for random variables. If X and Y have joint pdf $f_{X,Y}(x, y)$, then the conditional density of X given $Y = y$ is

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x, y)}{f_Y(y)}.$$

Thus, two random variables X and Y are independent if and only if

$$f_{X,Y}(x, y) = f_X(x)f_Y(y).$$

Also, if X and Y are independent, then

$$f_{X|Y=y}(x) \underset{\text{def'n}}{=} \frac{f_{X,Y}(x,y)}{f_Y(y)} \underset{\text{ind.}}{=} \frac{f_X(x)\cancel{f_Y(y)}}{\cancel{f_Y(y)}} = f_X(x).$$

4 Properties of Expected Value and Variance

Suppose that X and Y are random variables, and a and b are constants. Then the following hold:

1. $E[aX + b] =$ $a EX + b$

2. $E[X + Y] =$ $EX + EY$

3. If X and Y are independent, then $E[XY] =$ $E[X]\,E[Y]$.

4. $Var[b] =$ 0

5. $Var[aX + b] =$ $a^2 \, Var\, X$

6. If X and Y are independent, $Var[X + Y] =$ $Var X + Var Y$

5 Random Samples

Definition 5.1 Random variables $\{X_1, \ldots, X_n\}$ are defined as a *random sample* from f_X if $X_1, \ldots, X_n \overset{iid}{\sim} f_X$. "independent and identically distributed"

Example 5.1

$X_1, \ldots, X_n \overset{iid}{\sim} N(0, \sigma^2)$. vs.

$$\left.\begin{array}{l} X_1 \sim N(\mu_1, \sigma^2) \\ X_2 \sim N(\mu_2, \sigma^2) \end{array}\right\}$$ may be independent but NOT distributed identially. (not a random sample).

Theorem 5.1 If $X_1, \ldots, X_n \overset{iid}{\sim} f_X$, then

$$f_{X_1 \ldots X_n}(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_X(x_i).$$

joint pdf

product of marginals, easier to work with.

Example 5.2 Let $X_1, \ldots, X_n$ be iid. Derive the expected value and variance of the sample mean $\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

X_i's iid $\Rightarrow EX_1 = EX_2 = \ldots = EX_n$

$$E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \overset{\text{prop 1}}{=} \frac{1}{n} E\left[\sum_{i=1}^{n} X_i\right] \overset{\text{prop 2}}{=} \frac{1}{n}\sum_{i=1}^{n} EX_i = \frac{1}{n}\sum_{i=1}^{n} EX_1 = \frac{1}{\cancel{n}}\cancel{n}\, EX_1 = EX_1$$

X_i's iid $\Rightarrow \operatorname{Var}X_1 = \ldots = \operatorname{Var}X_n$

$$\operatorname{Var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \overset{\text{prop 5}}{=} \frac{1}{n^2}\operatorname{Var}\left[\sum X_i\right] \overset{\text{independence + prop 6}}{=} \frac{1}{n^2}\sum \operatorname{Var}X_i = \frac{1}{n^2}\sum \operatorname{Var}X_1 = \frac{1}{n^{\cancel{2}}}\cancel{n}\operatorname{Var}X_1 = \frac{\operatorname{Var}X_1}{n}$$

6 R Tips

From here on in the course we will be dealing with a lot of **randomness**. In other words, running our code will return a **random** result.

> But what about reproducibility??

When we generate "random" numbers in R, we are actually generating numbers that *look* random, but are *pseudo-random* (not really random). The vast majority of computer languages operate this way.

This means all is not lost for reproducibility!

```
set.seed(400)
```

Before running our code, we can fix the starting point (`seed`) of the pseudorandom number generator so that we can reproduce results.

Speaking of generating numbers, we can generate numbers (also evaluate densities, distribution functions, and quantile functions) from named distributions in R.

which distribution

randomly generate

evaluate densities

evaluate cdf

evaluate quantile

```
rnorm(100)
dnorm(x)
pnorm(x)
qnorm(y)
```

μ, σ?

may be useful to you for future homework...