2 Monte Carlo Methods for Hypothesis Tests

There are two aspects of hypothesis tests that we will investigate through the use of Monte Carlo methods: Type I error and Power.

Example 2.1 Assume we want to test the following hypotheses

$$H_0 : \mu = 5$$
$$H_a : \mu > 5$$

with the test statistic

$$T^* = \frac{\overline{x} - 5}{s/\sqrt{n}}.$$

This leads to the following decision rule:

critical value (quantile)

Reject H_0 if $T^* > t_{(1-\alpha/2), n-1} = qt(1 - \alpha/2, n-1)$. (in R).

equivalent to: Reject H_0 if p-value $< \alpha$.

What are we assuming about X?

$$X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$$

unknown.

2.1 Types of Errors

Type I error: Reject H_0 when H_0 true.

Type II error: Fail to reject H_0 when H_0 false.

Decision \ truth	H_0 true	H_0 false
Reject H_0	Type I error α	correct decision, power $= 1-\beta$
Fail to Reject H_0	correct decision	Type II error β

$\alpha = P(\text{reject } H_0 \mid H_0 \text{ true})$
$= P(\text{type I error})$

$\beta = P(\text{Fail to reject } H_0 \mid H_0 \text{ False})$
$= P(\text{type II error}).$

Usually we set $\alpha = 0.05$ or 0.10, and choose a sample size large enough such that power $= 1 - \beta \geq 0.80$.

For simple cases, we can find formulas for α and β.

For all others, we can use monte Carlo integration to estimate α & $1-\beta$
P(type I error) power.

2.2 MC Estimator of α

Assume $X_1, \dots, X_n \sim F(\theta_0)$ (i.e., assume H_0 is true).

Then, we have the following hypothesis test –

$$H_0 : \theta = \theta_0 \quad \text{a number}$$
$$H_a : \theta > \theta_0$$

and the statistics T^*, which is a test statistic computed from data. Then we **reject** H_0 if $T^* >$ the critical value from the distribution of the test statistic. assuming H_0 true.

This leads to the following algorithm to estimate the Type I error of the test (α) ← get $\hat{\alpha}$

For replicate $j = 1, \dots, m$

1. Generate $X_1^{(j)}, \dots, X_n^{(j)} \sim F(\theta_0)$
2. Compute $T^{*(j)} = \varphi\left(X_1^{(j)}, \dots, X_n^{(j)}\right)$ ← function of the data
3. Let $I_j = \begin{cases} 1 & \text{if reject } H_0 \text{ based on } T^{*(j)} \\ 0 & \text{o.w.} \end{cases}$

Then $\hat{\alpha} = \underbrace{\frac{1}{m} \sum_{j=1}^{m} I_j}_{\text{monte carlo estimate of an integral} = P(\text{rejecting } H_0 \mid H_0 \text{ true})} =$ estimated type I error $\left(\hat{P}(\text{rejecting } H_0 \mid H_0 \text{ true})\right)$

and $\hat{se}(\hat{\alpha}) = \sqrt{\frac{\hat{\alpha}(1-\hat{\alpha})}{m}}$ = estimate of $\sqrt{Var(\hat{\alpha})}$ = estimated uncertainty about estimate of α.

Why? $Var(\hat{\alpha}) \overset{\text{indep.}}{=} \frac{1}{m^2} \sum_{j=1}^{m} Var\, I_j$ and $I_j \sim Bernoulli(p)$ where

$p = P(\text{reject } H_0 \mid X_1, \dots, X_n \sim F(\theta_0)) = \alpha$.

$\Rightarrow Var\, I_j = \alpha(1-\alpha)$.

$\Rightarrow Var(\hat{\alpha}) = \frac{1}{m} \alpha(1-\alpha)$.

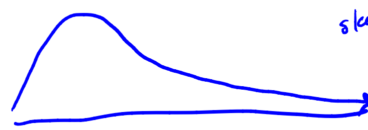
Your Turn

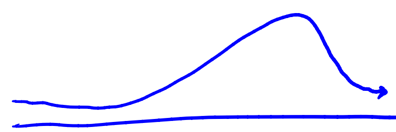
Example 2.2 (Pearson's moment coefficient of skewness) Let $X \sim F$ where $E(X) = \mu$ and $Var(X) = \sigma^2$. Let

$$\sqrt{\beta_1} = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right].$$

Then for a

- symmetric distribution, $\sqrt{\beta_1} = 0$,
- positively skewed distribution, $\sqrt{\beta_1} > 0$, and
- negatively skewed distribution, $\sqrt{\beta_1} < 0$.





The following is an estimator for skewness

$$\widehat{\sqrt{\beta_1}} = \sqrt{b_1} = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^3}{\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2\right]^{3/2}}.$$

It can be shown by Statistical theory that if $X_1, \dots, X_n \sim N(\mu, \sigma^2)$, then as $n \to \infty$,

$$\sqrt{b_1} \overset{\text{asymptotic}}{\boxed{\dot\sim}} N\left(0, \frac{6}{n}\right).$$

* Thus we can test the following hypothesis

$$\text{Two-sided test.}\left\{\begin{array}{l} H_0 : \sqrt{\beta_1} = 0 \leftarrow H_0\text{: symmetric distribution} \\ H_a : \sqrt{\beta_1} \neq 0 \end{array}\right.$$

by comparing $T^* = \boxed{\frac{\sqrt{b_1}}{\sqrt{\frac{6}{n}}}}$ to a critical value from a $\underline{N(0,1)}$ distribution.

In practice, convergence of $\sqrt{b_1}$ to a $N\left(0, \frac{6}{n}\right)$ is slow.

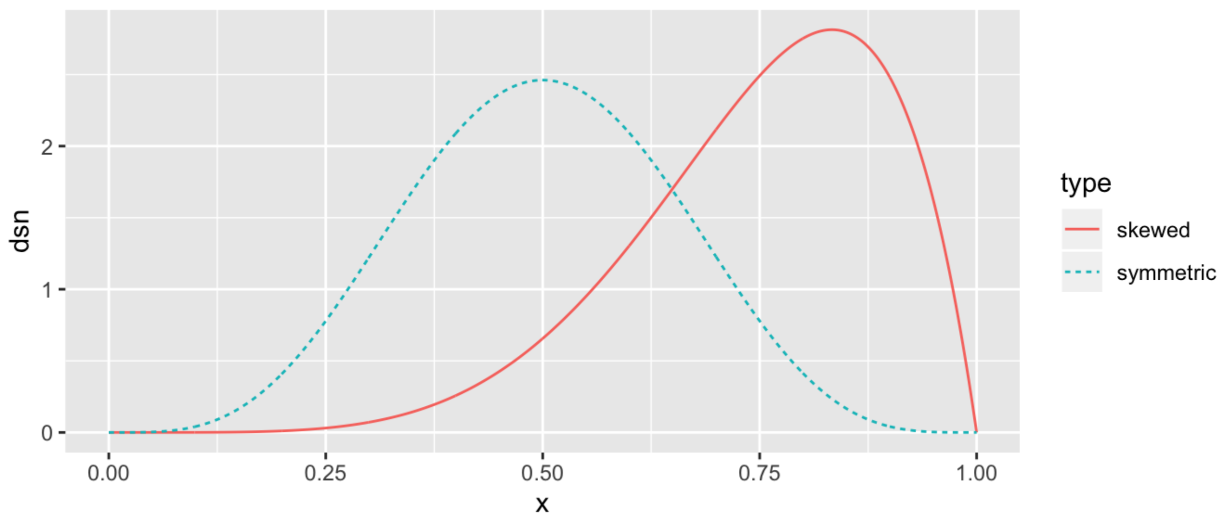
$\Rightarrow$ n needs to be large for dist of $\sqrt{b_1} \approx$ Normal.

We want to assess $P(\text{Type I error})$ for $\overset{\text{nominal}}{\alpha} = 0.05$ for $n = 10, 20, 30, 50, 100, 500$.

empirical Type I error $\hat{\alpha}$

```r
library(tidyverse)

# compare a symmetric and skewed distribution
data.frame(x = seq(0, 1, length.out = 1000)) %>%
  mutate(skewed = dbeta(x, 6, 2),
         symmetric = dbeta(x, 5, 5)) %>%
  gather(type, dsn, -x) %>%
  ggplot() +
  geom_line(aes(x, dsn, colour = type, lty = type))
```




```r
## write a skewness function based on a sample x
skew <- function(x) {
  YOUR TURN
}
```

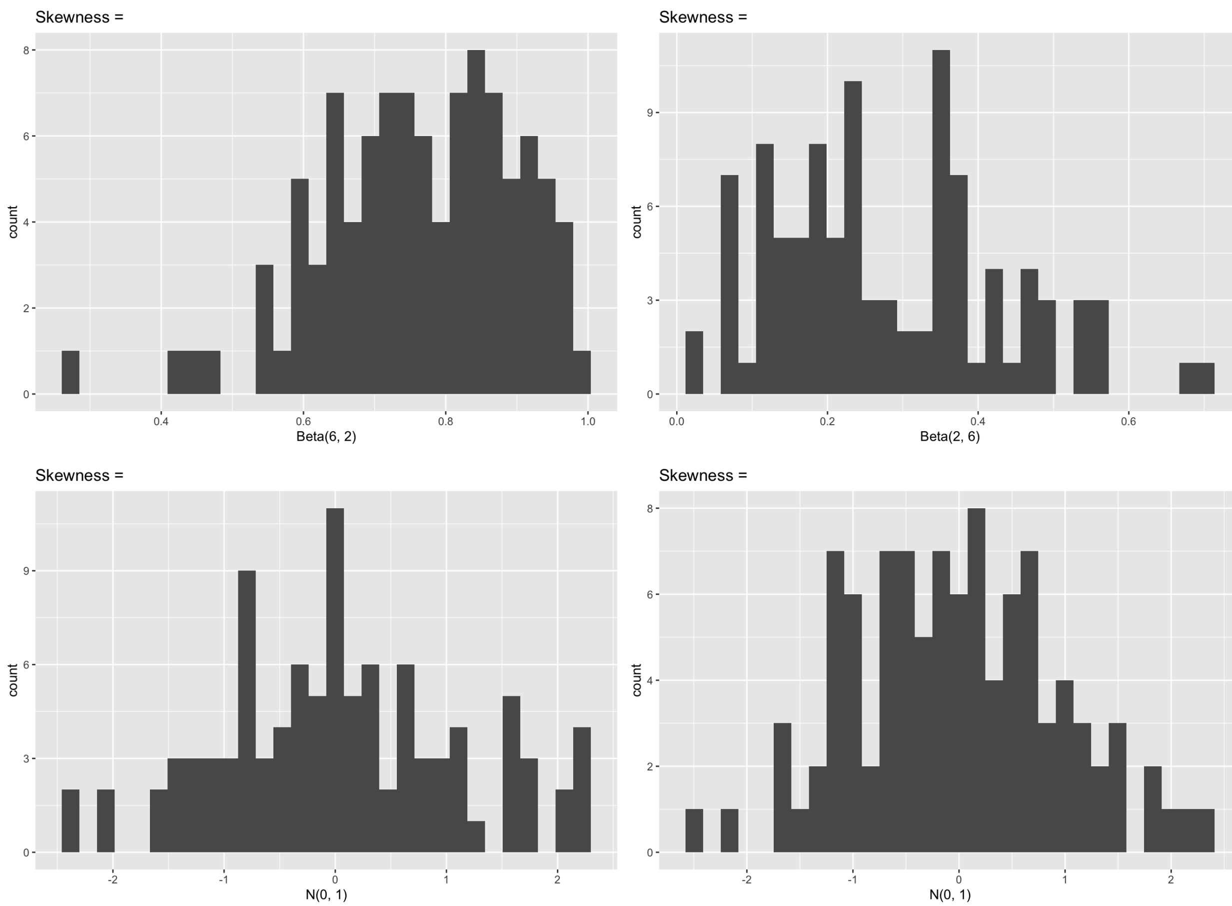
$$\sqrt{b_1} = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^3}{\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right]^{3/2}}$$

```r
## check skewness of some samples
n <- 100
a1 <- rbeta(n, 6, 2)
a2 <- rbeta(n, 2, 6)

## two symmetric samples
b1 <- rnorm(100)
b2 <- rnorm(100)

## fill in the skewness values
ggplot() + geom_histogram(aes(a1)) + xlab("Beta(6, 2)") +
  ggtitle(paste("Skewness = "))
```

```r
ggplot() + geom_histogram(aes(a2)) + xlab("Beta(2, 6)") +
  ggtitle(paste("Skewness = "))
ggplot() + geom_histogram(aes(b1)) + xlab("N(0, 1)") +
  ggtitle(paste("Skewness = "))
ggplot() + geom_histogram(aes(b2)) + xlab("N(0, 1)") +
  ggtitle(paste("Skewness = "))
```




```r
## Assess the P(Type I Error) for alpha = .05, n = 10, 20, 30, 50,
  100, 500
```

assume sampling from Beta dsn

write a function that takes in α, n, m, a, b ← params for beta dsn. returns $\hat{\alpha}$

Beta(5,5).

Example 2.3 (Pearson's moment coefficient of skewness with variance correction) One way to improve performance of this statistic is to adjust the variance for small samples. It can be shown that

$$Var(\sqrt{b_1}) = \frac{6(n-2)}{(n+1)(n+3)}.$$

Assess the Type I error rate of a skewness test using the finite sample correction variance.