

## 2 Monte Carlo Methods for Hypothesis Tests

There are two aspects of hypothesis tests that we will investigate through the use of Monte Carlo methods: Type I error and Power.

**Example 2.1** Assume we want to test the following hypotheses

$$H_0 : \mu = 5$$

$$H_a : \mu > 5$$

with the test statistic

$$T^* = \frac{\bar{x} - 5}{s/\sqrt{n}}.$$

This leads to the following decision rule:

Reject  $H_0$  if  $T^* > \bar{t}_{(1-\alpha/2), n-1}$  ← critical value (quantile)  $= qt(1 - \alpha/2, n-1)$ . (in R).

equivalent to: Reject  $H_0$  if p-value  $< \alpha$ .

What are we assuming about  $X$ ?

$$X_1, \dots, X_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$$

↑  
unknown.

### 2.1 Types of Errors

Type I error: Reject  $H_0$  when  $H_0$  true.

Type II error: Fail to reject  $H_0$  when  $H_0$  false.

	truth	
	$H_0$ true	$H_0$ false
Decision	Reject $H_0$ Type I error $\alpha$	correct decision power = $1 - \beta$
	Fail to Reject $H_0$ correct decision	Type II error $\beta$

$$\alpha = P(\text{reject } H_0 \mid H_0 \text{ true})$$

$$= P(\text{type I error})$$

$$\beta = P(\text{Fail to reject } H_0 \mid H_0 \text{ False})$$

$$= P(\text{type II error}).$$

Usually we set  $\alpha = 0.05$  or  $0.10$ , and choose a sample size <sup>large enough</sup> such that power =  $1 - \beta \geq 0.80$ .

For simple cases, we can find formulas for  $\alpha$  and  $\beta$ .

For all others, we can use monte carlo integration to estimate  $\alpha$  &  $1 - \beta$   
 $\rho(\text{type I error})$       power.

## 2.2 MC Estimator of $\alpha$

Assume  $X_1, \dots, X_n \sim F(\theta_0)$  (i.e., assume  $H_0$  is true).

Then, we have the following hypothesis test –

$$\begin{aligned} H_0 : \theta &= \theta_0 \\ H_a : \theta &> \theta_0 \end{aligned}$$

*a number* (pointing to  $\theta_0$ )

and the statistics  $T^*$ , which is a test statistic computed from data. Then we **reject**  $H_0$  if  $T^* >$  the critical value from the distribution of the test statistic. *assuming  $H_0$  true.*

This leads to the following algorithm to estimate the Type I error of the test ( $\alpha$ )  $\leftarrow$  get  $\hat{\alpha}$

For replicate  $j = 1, \dots, m$

1. Generate  $X_1^{(j)}, \dots, X_n^{(j)} \sim F(\theta_0)$

2. Compute  $T^{*(j)} = \Psi(X_1^{(j)}, \dots, X_n^{(j)}) \leftarrow$  function of the data

3. Let  $I_j = \begin{cases} 1 & \text{if reject } H_0 \text{ based on } T^{*(j)} \\ 0 & \text{o.w.} \end{cases}$

Then  $\hat{\alpha} = \frac{1}{m} \sum_{j=1}^m I_j =$  estimated type I error ( $\hat{p}(\text{rejecting } H_0 | H_0 \text{ true})$ )

*monte carlo estimate of an integral =  $P(\text{rejecting } H_0 | H_0 \text{ true})$ .*

and  $sc(\hat{\alpha}) = \sqrt{\frac{\hat{\alpha}(1-\hat{\alpha})}{m}}$  = estimate of  $\sqrt{\text{Var}(\hat{\alpha})}$  = estimated uncertainty about estimate of  $\alpha$ .

Why?  $\text{Var}(\hat{\alpha}) = \frac{1}{m^2} \sum_{j=1}^m \text{Var } I_j$  and  $I_j \sim \text{Bernoulli}(p)$  where

$$p = P(\text{reject } H_0 | X_1, \dots, X_n \sim F(\theta_0)) = \alpha.$$

$$\Rightarrow \text{Var}(\hat{\alpha}) = \frac{1}{m} \alpha(1-\alpha).$$

$$\Rightarrow \text{Var } I_j = \alpha(1-\alpha).$$

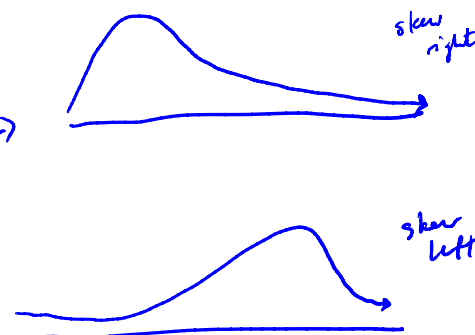
## Your Turn

**Example 2.2 (Pearson's moment coefficient of skewness)** Let  $X \sim F$  where  $E(X) = \mu$  and  $Var(X) = \sigma^2$ . Let

$$\sqrt{\beta_1} = E \left[ \left( \frac{X - \mu}{\sigma} \right)^3 \right].$$

Then for a

- symmetric distribution,  $\sqrt{\beta_1} = 0$ ,
- positively skewed distribution,  $\sqrt{\beta_1} > 0$ , and
- negatively skewed distribution,  $\sqrt{\beta_1} < 0$ .



The following is an estimator for skewness

$$\hat{\sqrt{\beta_1}} = \sqrt{b_1} = \frac{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^3}{\left[ \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 \right]^{3/2}}.$$

It can be shown by Statistical theory that if  $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ , then as  $n \rightarrow \infty$ ,

$$\sqrt{b_1} \overset{\text{asymptotic}}{\sim} N\left(0, \frac{6}{n}\right).$$

Thus we can test the following hypothesis

$$\begin{aligned} H_0 : \sqrt{\beta_1} &= 0 \leftarrow H_0: \text{symmetric distribution} \\ H_a : \sqrt{\beta_1} &\neq 0 \end{aligned}$$

by comparing  $\frac{\sqrt{b_1}}{\sqrt{\frac{6}{n}}}$  to a critical value from a  $N(0, 1)$  distribution.

In practice, convergence of  $\sqrt{b_1}$  to a  $N\left(0, \frac{6}{n}\right)$  is slow.

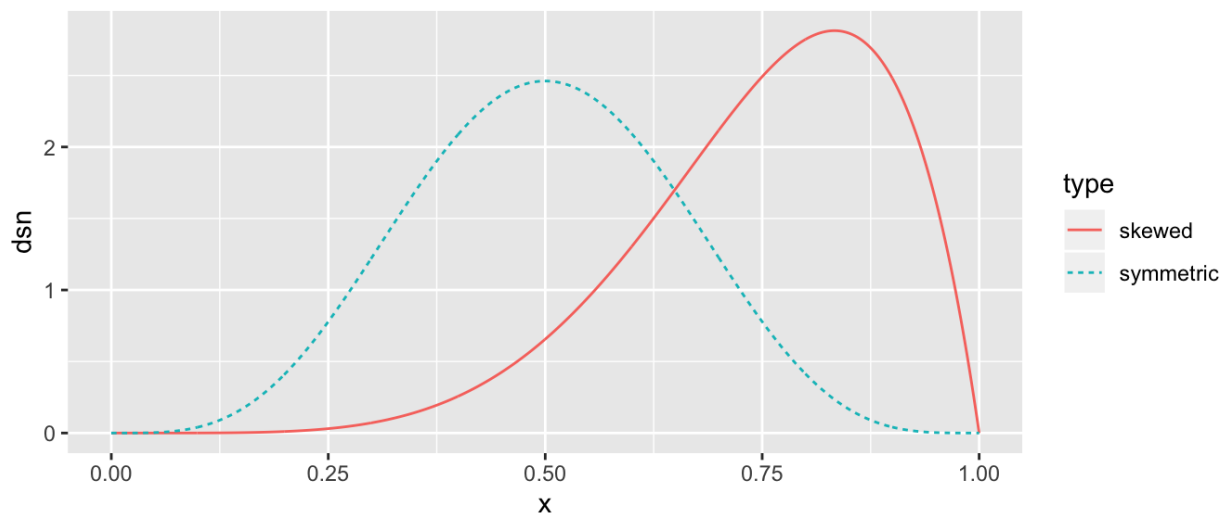
$\Rightarrow n$  needs to be large for dist of  $\sqrt{b_1} \approx \text{Normal}$ .

We want to assess  $P(\text{Type I error})$  for  $\alpha = 0.05$  for  $n = 10, 20, 30, 50, 100, 500$ .

$\hat{\alpha}$   
empirical Type I error

```
library(tidyverse)

# compare a symmetric and skewed distribution
data.frame(x = seq(0, 1, length.out = 1000)) %>%
  mutate(skewed = dbeta(x, 6, 2),
         symmetric = dbeta(x, 5, 5)) %>%
  gather(type, dsn, -x) %>%
  ggplot() +
  geom_line(aes(x, dsn, colour = type, lty = type))
```



```
## write a skewness function based on a sample x
skew <- function(x) {
}
```

$$\leftarrow \sqrt{b_1} = \frac{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^3}{\left[ \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \right]^{3/2}}$$

```
## check skewness of some samples
n <- 100
a1 <- rbeta(n, 6, 2)
a2 <- rbeta(n, 2, 6)

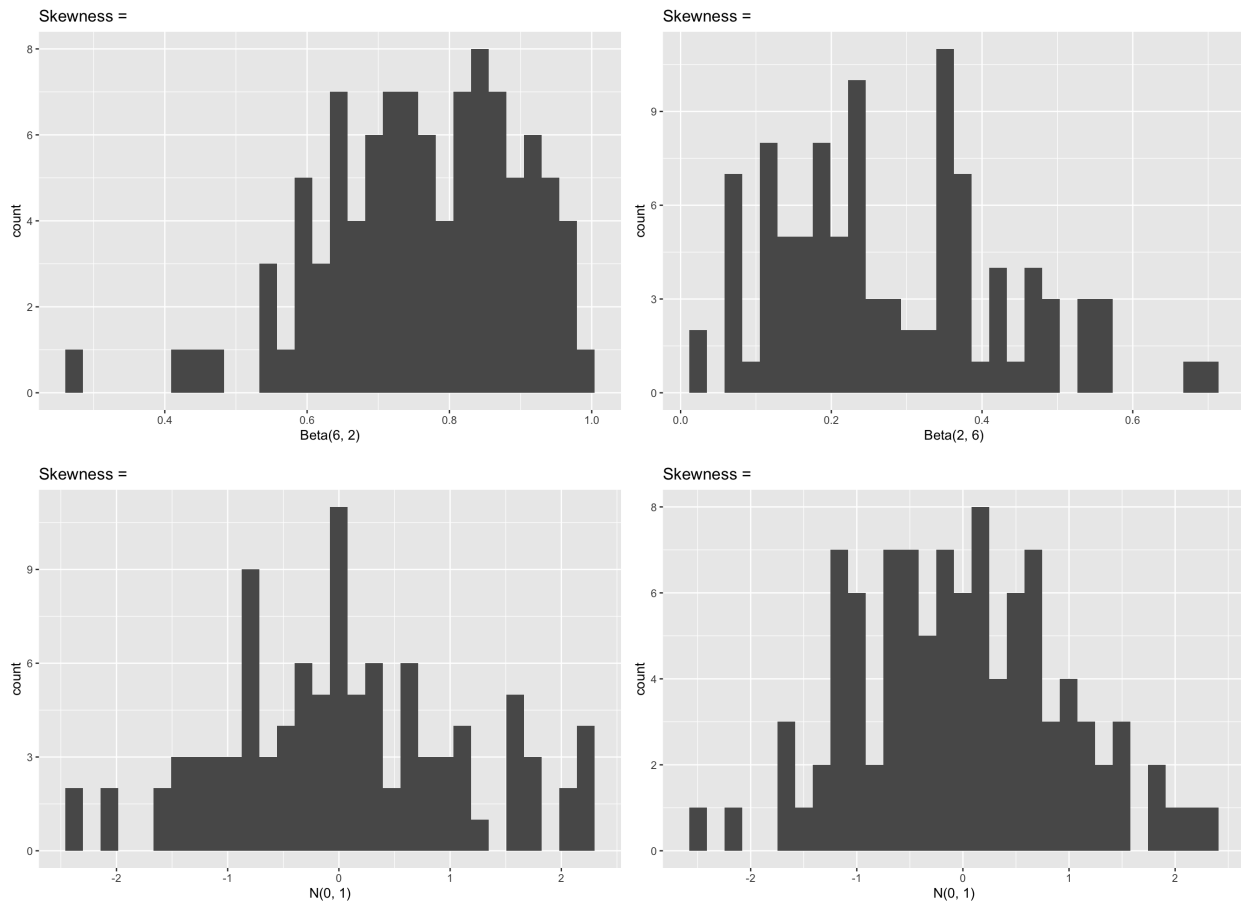
## two symmetric samples
b1 <- rnorm(100)
b2 <- rnorm(100)
```

```
## fill in the skewness values
ggplot() + geom_histogram(aes(a1)) + xlab("Beta(6, 2)") +
  ggtitle(paste("Skewness = "))
```

```

ggplot() + geom_histogram(aes(a2)) + xlab("Beta(2, 6)") +
  ggtitle(paste("Skewness = "))
ggplot() + geom_histogram(aes(b1)) + xlab("N(0, 1)") +
  ggtitle(paste("Skewness = "))
ggplot() + geom_histogram(aes(b2)) + xlab("N(0, 1)") +
  ggtitle(paste("Skewness = "))

```



## Assess the  $P(\text{Type I Error})$  for  $\alpha = .05$ ,  $n = 10, 20, 30, 50, 100, 500$

**Example 2.3 (Pearson's moment coefficient of skewness with variance correction)** One way to improve performance of this statistic is to adjust the variance for small samples. It can be shown that

$$\text{Var}(\sqrt{b_1}) = \frac{6(n-2)}{(n+1)(n+3)}.$$

Assess the Type I error rate of a skewness test using the finite sample correction variance.