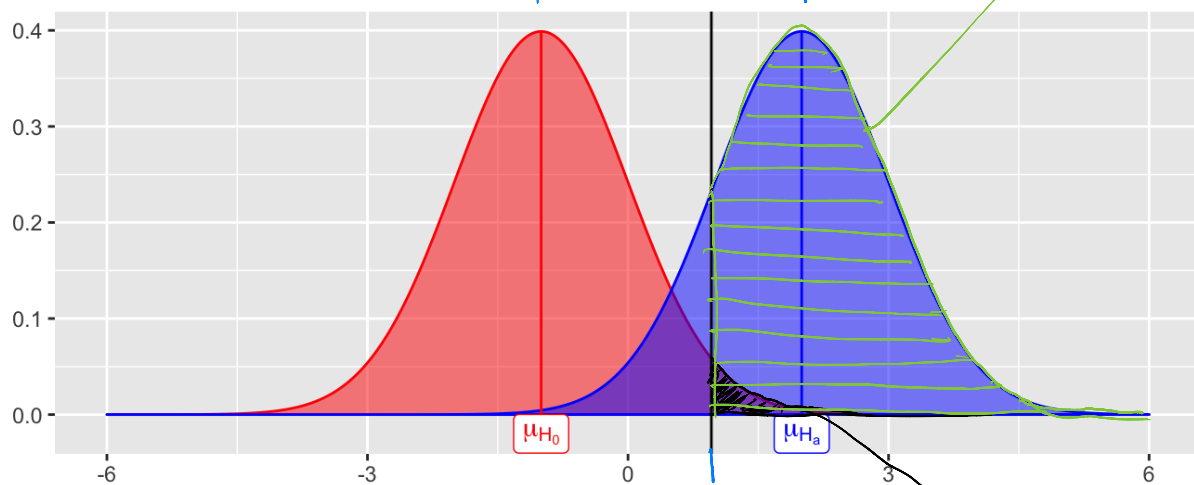
2.3 Power

effect size

$1-\beta = \text{Power}$



$= P(\text{type I error}) = \alpha$

decision criterion

fixed by setting α

critical value $=$ qnorm$(1-\alpha, \mu_{H_0}, \sigma)$

Consider a hypothesis test about the parameter θ:

$$H_0 : \theta = \theta_0$$
$$H_a : \theta > \theta_0$$

We let $\beta = P(\text{fail to reject } H_0 | H_0 \text{ is false}) = P(\text{Type II error})$, then Power $=$ $P(\text{reject } H_0 | H_0 \text{ is false}) = 1 - \beta$.

Power depends on the distance between the hypothesized value of the parameter θ_0 and the actual value θ_1, so we can write $1 - \beta(\theta_1)$.

↳ effect size.

Why is power important?

1. If you have multiple statistical testing method for the same hypothesis, choose the test that is the most powerful.

2. If you're going to spend time/money to do an experiment, need to check beforehand that your study will be powerful enough to detect an effect.

For a few simple cases, you can derive a closed form expression of power.

All others: use Monte Carlo methods to estimate power.

Example 2.4 Consider a one-sample z-test. Sample $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$.

μ unknown, σ^2 known.

$H_0: \mu = \mu_0$ vs. $H_a: \mu > \mu_0$

Using statistic $z^* = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$, we reject H_0 if $z^* > z_{1-\alpha}$

If $\mu_0 = 5$ (hypothesized value) but the true mean is $\mu_1 = 6$.
What is the prob. of correctly rejecting $H_0: \mu = 5$? This is power.

Effect size: $\mu_1 - \mu_0 = 6 - 5 = 1$. If the effect size was 10, our test would have more power (easier to detect the truth).

For the z-test, we can derive power (Chihara & Hesterberg p. 229–230).

$$1 - \beta = P(\text{reject } H_0 \mid H_0 \text{ is false})$$
$$= P\Big(z^* > \underbrace{z_{1-\alpha} - \frac{(\mu_1 - \mu_0)}{\sigma/\sqrt{n}}}_{\text{smallest } z \text{ where you can reject } H_0}\Big)$$

So power is a function of

1. Significance level: as $\alpha \uparrow$, power $\uparrow$ [trade off btw type I and type II error]
2. Effect size: $\mu_1 - \mu_0$ as effect size $\uparrow$, power $\uparrow$
3. Sample size: as $n \uparrow$, power $\uparrow$
4. Variance: as variance $\downarrow$, power $\uparrow$ (no control over this in practice).

Notes: (1) as power $= 1 - \beta \uparrow$, $P(\text{type I error}) = \alpha \uparrow$. For fixed n, σ, & $\mu_1 - \mu_0$, the only way to increase power, is to $\uparrow \alpha$.

(2) The only way to simultaneously $\uparrow$ power & $\downarrow \alpha$, must $\uparrow n$.

2.4 MC Estimator of $1 - \beta$

Assume $X_1, \ldots, X_n \sim F(\theta_0)$ (i.e., assume H_0 is true).

Then, we have the following hypothesis test –

$$H_0 : \theta = \theta_0$$
$$H_a : \theta > \theta_0$$

and the statistics T^*, which is a test statistic computed from data. Then we **reject** H_0 if $T^* >$ the critical value from the distribution of the test statistic.

This leads to the following algorithm to estimate the power of the test $(1 - \beta)$

1. Select model, setup hypothesis test.
2. Select value of alternative θ_1
3. Set n, other parameter values (e.g. 6), and α
4. Then for each $j = 1, \ldots, m$
   a) Sample $X_1^{(j)}, \ldots, X_n^{(j)}$ from the model under the alternative hypothesis $\theta = \theta_1$
   b) Compute $T^{*(j)}$ based on data from a). e.g. $\mathbb{1}\{T^{*(j)} > \text{crit. value}\}$ based on H_a.
   c) Compute $y_j = \mathbb{1}\{\text{reject } H_0 \text{ based } T^{*(j)}\}$
5. Compute $1 - \hat{\beta} = \frac{1}{m} \sum_{j=1}^{m} y_j$ (i.e. count of correct answers).

Your Turn

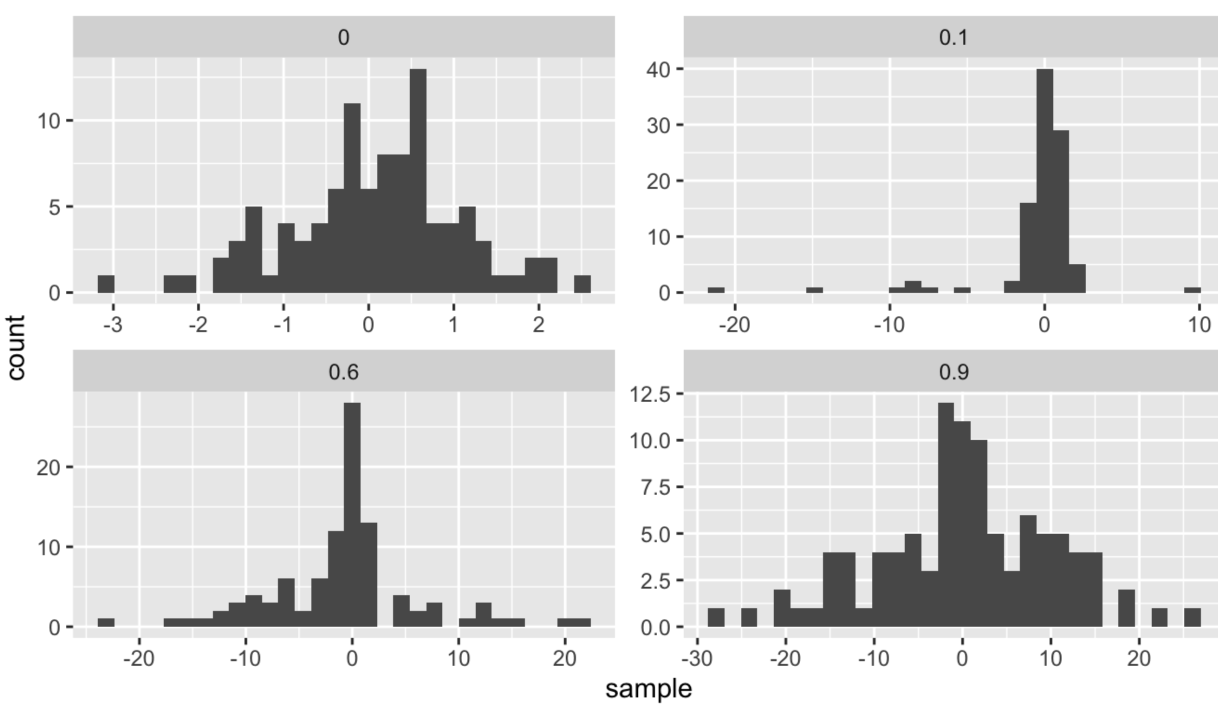
Consider data generated from the following mixture distribution:

$$f(x) = (1-\epsilon)f_1(x) + \epsilon f_2(x), \quad x \in \mathbb{R}$$

where f_1 is the pdf of a $N(0,1)$ distribution, f_2 is the pdf of a $N(0,100)$ distribution, and $\epsilon \in [0,1]$.

```
r_noisy_normal <- function(n, epsilon) {
  z <- rbinom(n, 1, 1 - epsilon)
  z*rnorm(n, 0, 1) + (1 - z)*rnorm(n, 0, 10)
}

n <- 100
data.frame(e = 0, sample = r_noisy_normal(n, 0)) %>%
  rbind(data.frame(e = 0.1, sample = r_noisy_normal(n, 0.1))) %>%
  rbind(data.frame(e = 0.6, sample = r_noisy_normal(n, 0.6))) %>%
  rbind(data.frame(e = 0.9, sample = r_noisy_normal(n, 0.9))) %>%
  ggplot() +
  geom_histogram(aes(sample)) +
  facet_wrap(.~e, scales = "free")
```



We will compare the power of various tests of normality. Let F_X be the distribution of a random variable X. We will consider the following hypothesis test,

$$H_0 : F_x \in N \qquad \text{vs.} \qquad H_a : F_x \notin N,$$

where N denotes the family of univariate Normal distributions.

Recall Pearson's moment coefficient of skewness (See Example 2.2).

We will compare Monte Carlo estimates of power for different levels of contamination ($0 \leq \epsilon \leq 1$). We will use $\alpha = 0.1$, $n = 100$, and $m = 100$.

```
# skewness statistic function
skew <- function(x) {
  xbar <- mean(x)
  num <- mean((x - xbar)^3)
  denom <- mean((x - xbar)^2)
  num/denom^1.5
}

# setup for MC
alpha <- .1
n <- 100
m <- 100
epsilon <- seq(0, 1, length.out = 200)
var_sqrt_b1 <- 6*(n - 2)/((n + 1)*(n + 3)) # adjusted variance for
  skewness test
crit_val <- qnorm(1 - alpha/2, 0, sqrt(var_sqrt_b1)) #crit value for
  the test
empirical_pwr <- rep(NA, length(epsilon)) #storage

# estimate power for each value of epsilon
for(j in 1:length(epsilon)) {
  # perform MC to estimate empirical power
  ## Your turn

}

## store empirical se
empirical_se <- "Your Turn: fill this in"

## plot results --
## x axis = epsilon values
## y axis = empirical power
## use lines + add band of estimate +/- se
```

We can detect contamination levels between .015 and .15 power ≥ 0.8 when $n = 100$ ⟶ ε is like effect size (distance from 0)

Compare the power with $n = 100$ to the power with $n = 10$. Make a plot to compare the two for many values of ϵ.

Recall power depends on 3 things:

1. level of the test α
2. sample size n
3. effect size

When $n=10$, power < 0.8 for all levels of ε