2 Monte Carlo Methods for Hypothesis Tests

There are two aspects of hypothesis tests that we will investigate through the use of Monte Carlo methods: Type I error and Power.

Example 2.1 Assume we want to test the following hypotheses

$$H_0 : \mu = 5$$
$$H_a : \mu > 5$$

with the test statistic

$$T^* = \frac{\overline{x} - 5}{s/\sqrt{n}}.$$

This leads to the following decision rule:

critical value.

Reject H_0 if $T^* > t_{(1-\alpha/2), n-1} = qt(1-\alpha/2, n-1)$

equivalent to: Reject H_0 if p-value $< \alpha$.

What are we assuming about X?

$X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

unknown

2.1 Types of Errors

Type I error: Reject H_0 when H_0 true

Type II error: Fail to reject H_0 when H_0 false.

Truth

Decision

	H_0 true	H_0 False
Reject H_0	Type I error α	correct decision Power $= 1-\beta$
Fail to reject H_0	correct decision	Type II error β

$\alpha = P(\text{reject } H_0 \mid H_0 \text{ true})$
$= P(\text{Type I error})$

$\beta = P(\text{Fail to reject } H_0 \mid H_0 \text{ False})$
$= P(\text{Type II error}).$

Usually we set $\alpha = 0.05$ or 0.10, and choose a sample size such that power $= 1 - \beta \geq 0.80$.

P(type I error) power

For simple cases, we can find formulas for α and β.

For all others, we can use Monte Carlo integration to estimate α and $1-\beta$

2.2 MC Estimator of α

Assume $X_1, \ldots, X_n \sim F(\theta_0)$ (i.e., assume H_0 is true).

Then, we have the following hypothesis test –

$$H_0 : \theta = \theta_0$$
$$H_a : \theta > \theta_0$$

and the statistics T^*, which is a test statistic computed from data. Then we **reject** H_0 if $T^* >$ the critical value from the distribution of the test statistic.

This leads to the following algorithm to estimate the Type I error of the test (α) get $\hat{\alpha}$

For replicate $j = 1, \ldots, m$

null distribution.

1. Generate $X_1^{(j)}, \ldots, X_n^{(j)} \sim F(\theta_0)$
2. Compute $T^{*(j)} = \Upsilon(X_1^{(j)}, \ldots, X_n^{(j)})$ where Υ is a function of data.
3. Let $I_j = \begin{cases} 1 & \text{if reject } H_0 \text{ based on } T^{*(j)} \\ 0 & \text{o.w.} \end{cases}$

Then $\hat{\alpha} = \frac{1}{m}\sum_{i=1}^{m} I_j =$ estimate type I error rate $(\hat{P}(\text{reject } H_0 \mid H_0 \text{ true}))$

and $\hat{se}(\hat{\alpha}) = \sqrt{\frac{\hat{\alpha}(1-\hat{\alpha})}{m}} =$ estimate of $\sqrt{Var(\hat{\alpha})}$ = estimated uncertainty about estimator of α.

why? $Var(\hat{\alpha}) = \frac{1}{m^2}\sum_{i=1}^{m} Var\, I_j$, and $I_j \overset{iid}{\sim} \text{Bernoulli}(p)$, where

$\Rightarrow Var\, I_j = \alpha(1-\alpha)$ $\quad p = P(\text{reject } H_0 \mid X_1, \ldots, X_n \sim F(\theta_0)) = \alpha$

$\Rightarrow Var(\hat{\alpha}) = \frac{1}{m}\alpha(1-\alpha)$

$\Rightarrow \hat{Var}(\hat{\alpha}) = \frac{1}{m}\hat{\alpha}(1-\hat{\alpha})$.

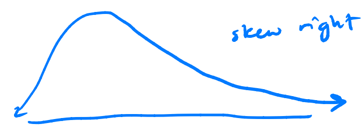
Your Turn

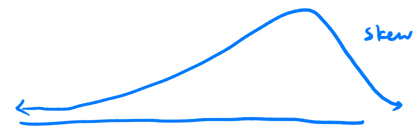
Example 2.2 (Pearson's moment coefficient of skewness) Let $X \sim F$ where $E(X) = \mu$ and $Var(X) = \sigma^2$. Let

$$\sqrt{\beta_1} = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right].$$

Then for a

- symmetric distribution, $\sqrt{\beta_1} = 0$,
- positively skewed distribution, $\sqrt{\beta_1} > 0$, and
- negatively skewed distribution, $\sqrt{\beta_1} < 0$.





The following is an estimator for skewness

$$\sqrt{b_1} = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^3}{\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2\right]^{3/2}}.$$

It can be shown by Statistical theory that if $X_1, \dots, X_n \sim N(\mu, \sigma^2)$, then as $n \to \infty$,

$$\sqrt{b_1} \mathrel{\dot\sim} N\left(0, \frac{6}{n}\right).$$

Thus we can test the following hypothesis

$$H_0 : \sqrt{\beta_1} = 0$$
$$H_a : \sqrt{\beta_1} \neq 0$$

H_0: symmetric dsn.

H_0: do. not.

by comparing $\frac{\sqrt{b_1}}{\sqrt{\frac{6}{n}}}$ to a critical value from a $N(0,1)$ distribution.

or we would compare $\sqrt{b_1}$ to a critical value from $N(0, \frac{6}{n})$.

In practice, convergence of $\sqrt{b_1}$ to a $N\left(0, \frac{6}{n}\right)$ is slow.

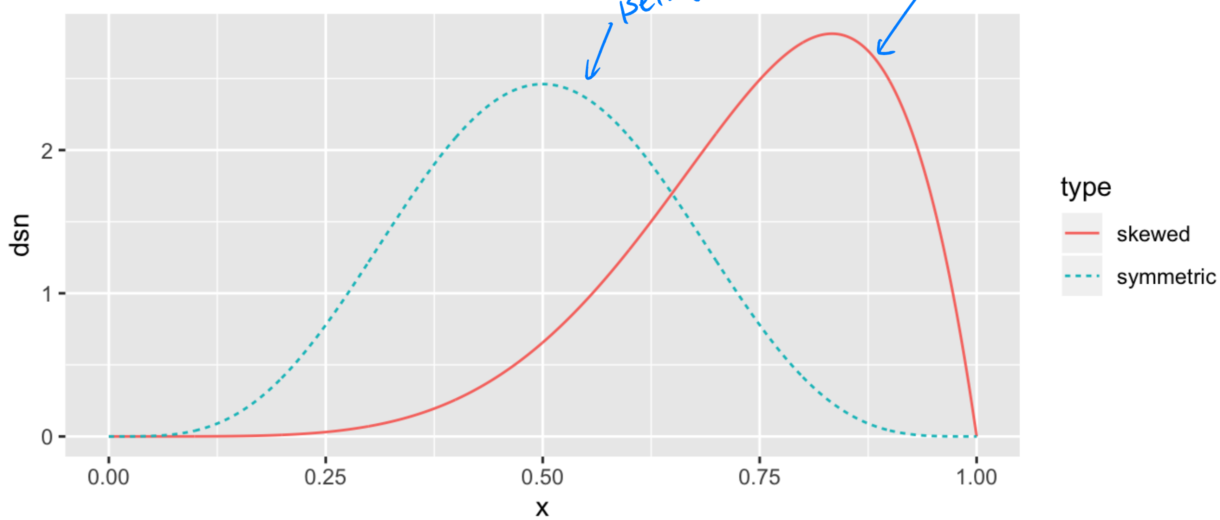
$\Rightarrow$ n needs to be large for dsn of $\sqrt{b_1}$ $\approx$ Normal.

We want to assess $P(\text{Type I error})$ for $\alpha = 0.05$ for $n = 10, 20, 30, 50, 100, 500$.

```
library(tidyverse)

# compare a symmetric and skewed distribution
data.frame(x = seq(0, 1, length.out = 1000)) %>%
  mutate(skewed = dbeta(x, 6, 2),
         symmetric = dbeta(x, 5, 5)) %>%
  gather(type, dsn, -x) %>%
  ggplot() +
  geom_line(aes(x, dsn, colour = type, lty = type))
```

Beta(6,2).

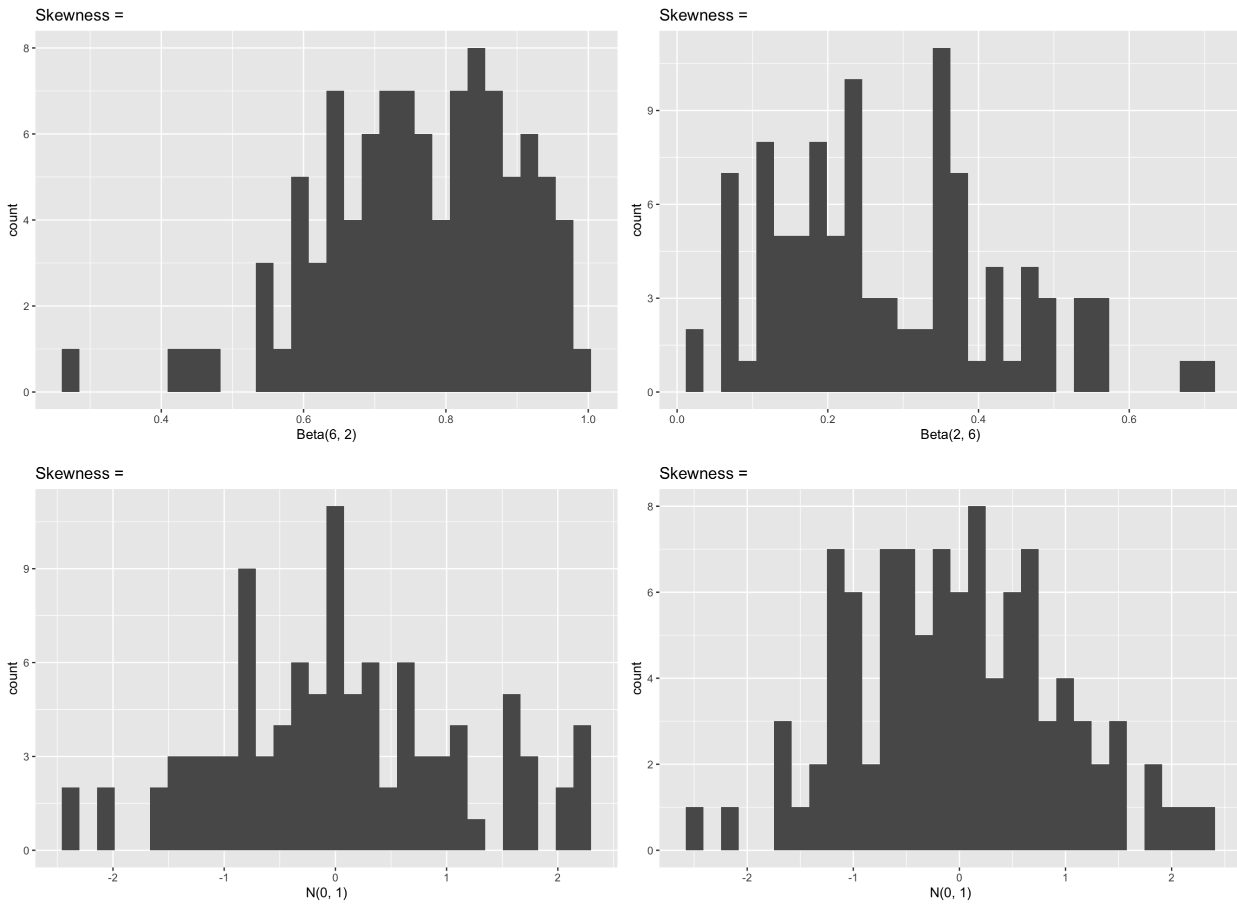
Beta(5,5)




```
## write a skewness function based on a sample x
skew <- function(x) {
        YOUR TURN (should return √b1)
}

## check skewness of some samples
n <- 100
a1 <- rbeta(n, 6, 2)
a2 <- rbeta(n, 2, 6)

## two symmetric samples
b1 <- rnorm(100)
b2 <- rnorm(100)

## fill in the skewness values
ggplot() + geom_histogram(aes(a1)) + xlab("Beta(6, 2)") +
  ggtitle(paste("Skewness = "))
```

add in skewness statistic

```r
ggplot() + geom_histogram(aes(a2)) + xlab("Beta(2, 6)") +
  ggtitle(paste("Skewness = "))
ggplot() + geom_histogram(aes(b1)) + xlab("N(0, 1)") +
  ggtitle(paste("Skewness = "))
ggplot() + geom_histogram(aes(b2)) + xlab("N(0, 1)") +
  ggtitle(paste("Skewness = "))
```




```r
## Assess the P(Type I Error) for alpha = .05, n = 10, 20, 30, 50,
  100, 500
```

Example 2.3 (Pearson's moment coefficient of skewness with variance correction) One way to improve performance of this statistic is to adjust the variance for small samples. It can be shown that

$$Var(\sqrt{b_1}) = \frac{6(n-2)}{(n+1)(n+3)}.$$

Assess the Type I error rate of a skewness test using the finite sample correction variance.