Chapter 2: Probability for Statistical Computing

We will **briefly** review some definitions and concepts in probability and statistics that will be helpful for the remainder of the class.

Just like we reviewed computational tools (`R` and packages), we will now do the same for probability and statistics.

Note: This is not meant to be comprehensive. I am assuming you already know this and maybe have forgotten a few things.

i.e. you may need to look some things up on your own.

https://xkcd.com/892/

Alternative text: "Hell, my eighth grade science class managed to conclusively reject it just based on a classroom experiment. It's pretty sad to hear about million-dollar research teams who can't even manage that."

1 Random Variables and Probability

Definition 1.1 A *random variable* is a function that maps sets of all possible outcomes of an experiment (sample space Ω) to $\mathbb{R}$.

"real numbers

$(-\infty, \infty)$

Example 1.1

experiment Toss 2 dice.

$\Omega = \{(i,j) : i = 1,..,6;\ j = 1, .., 6\}$

r.v. X = sum of the dice.

Example 1.2

experiment: Randomly select 25 deer & test for CWD

r.v. X_i $\{0 \text{ or } 1\}$ observe $X_1, ..., X_{25}$

$\Omega = \{+, - \text{ CWD}\}$

r.v. $p = \sum_{i=1}^{25} X_i / 25$ is also a r.v.!

Example 1.3

experiment: Deck of cards, draw one card

r.v. X : 1 if clubs, 0 otherwise.

$\Omega = \{$values of all 52 cards in a deck$\}$
$= \{AC, 2C, 3C, ..., KC,$
$AS, 2S, ..., KS,$
$AD, 2D, ..., KD,$
$AH, 2H, ..., KH\}$

Ex. 1.4

Today's high temp $= X_i$

Types of random variables –

Discrete take values in a countable set.

Ex. 1.1

X_i from Ex 1.2, X from X 1.3

Continuous take values in an uncountable set (like $\mathbb{R}$)

Ex 1.4 $X_i \in \mathbb{R}$

p from Ex 1.2, $p \in [0, 1]$

1.1 Distribution and Density Functions

Definition 1.2 The *probability mass function (pmf)* of a random variable X is f_X defined by

Notation: sometimes when the r.v. is obvious I will omit the subscript and write $f(x)$.

$$f_X(x) = P(X = x) \quad \text{for any } x \in \mathbb{R}$$

where $P(\cdot)$ denotes the probability of its argument.

There are a few requirements of a **valid** pmf

1. $f(x) \geq 0$ for all $x \in \mathbb{R}$.
2. $\sum_x f(x) = 1$
3. We call $\mathcal{X} = \{x : f(x) > 0\}$ the "support" of X.

Not a requirement

Example 1.4 Let Ω = all possible values of a roll of a single die $= \{1, \ldots, 6\}$ and X be the outcome of a single roll of one die $\in \{1, \ldots, 6\}$.

Fair die

$f(1) = P(X=1) = \frac{1}{6}$
$\vdots$
$f(6) = \frac{1}{6}$

$\sum_{x \in \mathcal{X}} f(x) = \sum_{x=1}^{6} \frac{1}{6} = 1$ ✓ valid pmf.

A pmf is defined for **discrete variables**, but what about **continuous**? Continuous variables do not have positive probability mass at any single point.

Definition 1.3 The *probability density function (pdf)* of a random variable X is f_X defined by

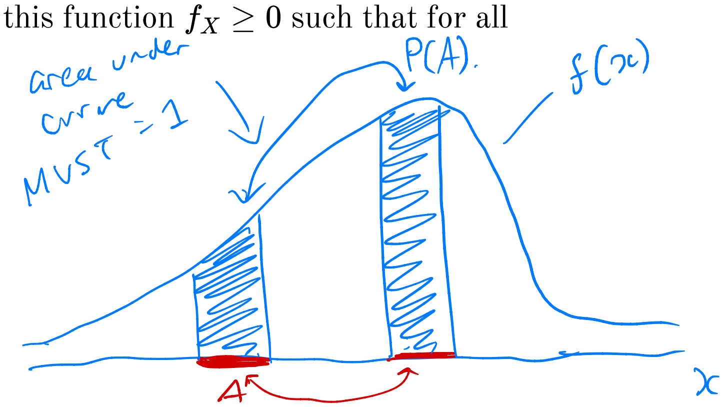
$A \subset \mathbb{R}$

$$P(X \in A) = \int_{x \in A} f_X(x)dx.$$

X is a continuous random variable if there exists this function $f_X \geq 0$ such that for all $x \in \mathbb{R}$, this probability exists.

For f_X to be a valid pdf,

1. $f(x) \geq 0 \quad \forall x \in \mathbb{R}$
2. $\int_{\mathbb{R}} f(x)dx = 1$



Again $\mathcal{X} = \{x : f(x) > 0\}$ is the "support" of X

There are many named pdfs and ~~cdfs~~ pmfs that you have seen in other class, e.g.

Bernoulli (pmf), Poisson (pmf), Gamma (pdf), Normal (pdf), Beta, exponential, hypergeometric.

Example 1.5 Let

$$f(x) = \begin{cases} c(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

← support

Find c and then find $P(X > 1)$

↳ such that $f(x)$ is a valid pdf.

$$1 = \int_0^2 c(4x - 2x^2)\,dx = c\left[2x^2 - \frac{2x^3}{3}\right]_0^2 = c\left[\frac{8}{3}\right] \Rightarrow c = \frac{3}{8}$$

"normalizing constant"

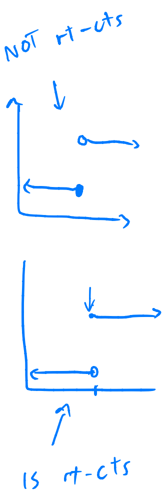
$$P(X \geq 1) = \int_1^\infty f(x)\,dx = \int_1^2 \frac{3}{8}(4x - 2x^2)\,dx = \frac{3}{8}\left[2x^2 - \frac{2x^3}{3}\right]_1^2 = \frac{1}{2}$$

Definition 1.4 The *cumulative distribution function (cdf)* for a random variable X is F_X defined by

cdf function of $x \in \mathbb{R}$

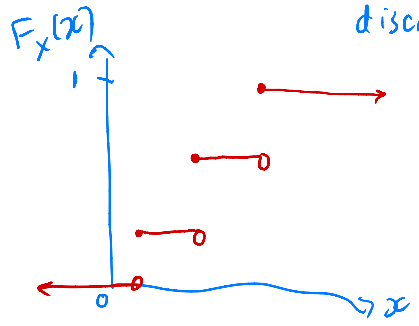
$$F_X(x) = P(X \leq x), \quad x \in \mathbb{R}.$$

↳ same def'n for both cts and discrete r.v.'s



The cdf has the following properties

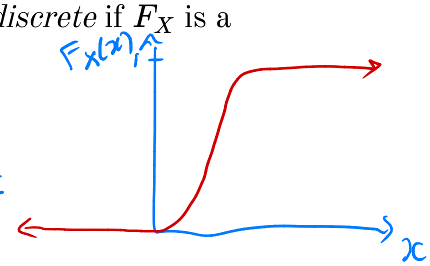
1. F_X is non-decreasing
2. F_X is right-continuous
3. $\lim_{x \to -\infty} F_X(x) = 0$ $\quad \lim_{x \to \infty} F_X(x) = 1$



"step function"

X is discrete

A random variable X is *continuous* if F_X is a continuous function and *discrete* if F_X is a step function.



Example 1.6 Find the cdf for the previous example.

$F_X(x) = P(X \leq x)$

for $x \in (0,2)$, $P(X \leq x) = \int_0^x \frac{3}{8}(4y - 2y^2)\,dy = \left[\frac{3}{8}\left(2y^2 - \frac{2y^3}{3}\right)\right]_0^x = \frac{3}{4}x^2\left(1 - \frac{x}{3}\right)$

So, $F_X(x) = \begin{cases} 0 & x \leq 0 \\ \frac{3}{4}x^2(1 - \frac{x}{3}) & x \in (0,2) \\ 1 & x \geq 2 \end{cases}$

Note $f(x) = F'(x) = \frac{dF(x)}{dx}$ in the continuous case.

pdf $\quad$ derivative of cdf

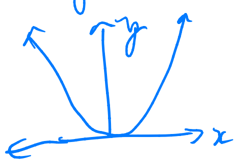
due to the Fundamental Th'm of Calculus.

Recall an indicator function is defined as

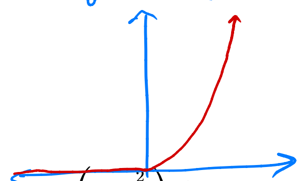
$$1_{\{A\}} = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{otherwise} \end{cases}.$$

condition

Example 1.7 $y = x^2$



$y = x^2 1\{x>0\}$

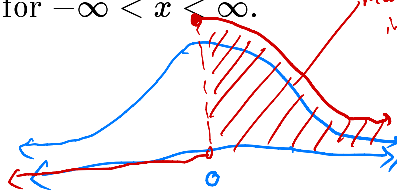


Example 1.8 If $X \sim N(0,1)$, the pdf is $f(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$ for $-\infty < x < \infty$.

YOUR TURN

If $f(x) = \frac{c}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\boxed{1_{\{x>0\}}}$, what is c?

must integrate to 1.



We have symmetry of $N(0,1)$ around 0.

$$\Rightarrow \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)dx = \frac{1}{2}$$

$$1 = c\int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)dx = \frac{c}{2} \Rightarrow c = 2.$$

need this to hold.

1.2 Two Continuous Random Variables

2 r.v.'s that we care about together and are both cts.

Definition 1.5 The *joint pdf* of the continuous vector (X, Y) is defined as

$$P((X,Y) \in A) = \iint_A f_{X,Y}(x,y)\,dx\,dy$$

joint pdf

for any set $A \subset \mathbb{R}^2$.

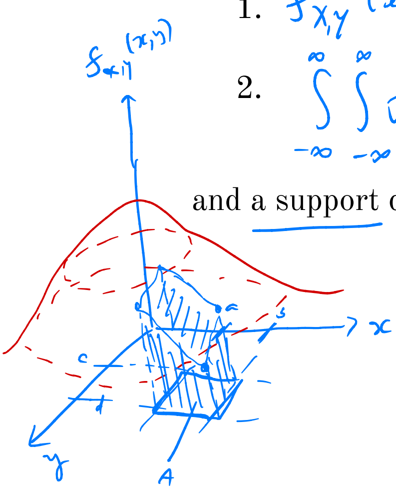
Joint pdfs have the following properties

1. $f_{X,Y}(x,y) \geq 0 \quad \forall\, x, y \in \mathbb{R}^2$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_X(x,y)\,dx\,dy = 1$

Note: We can also have jointly discrete r.v.'s w/ corresponding joint pmfs where

$$\sum\sum f_{X,Y}(x,y) = 1.$$

and a support defined to be $\{(x,y) : f_{X,Y}(x,y) > 0\}$. $= \mathcal{X}$



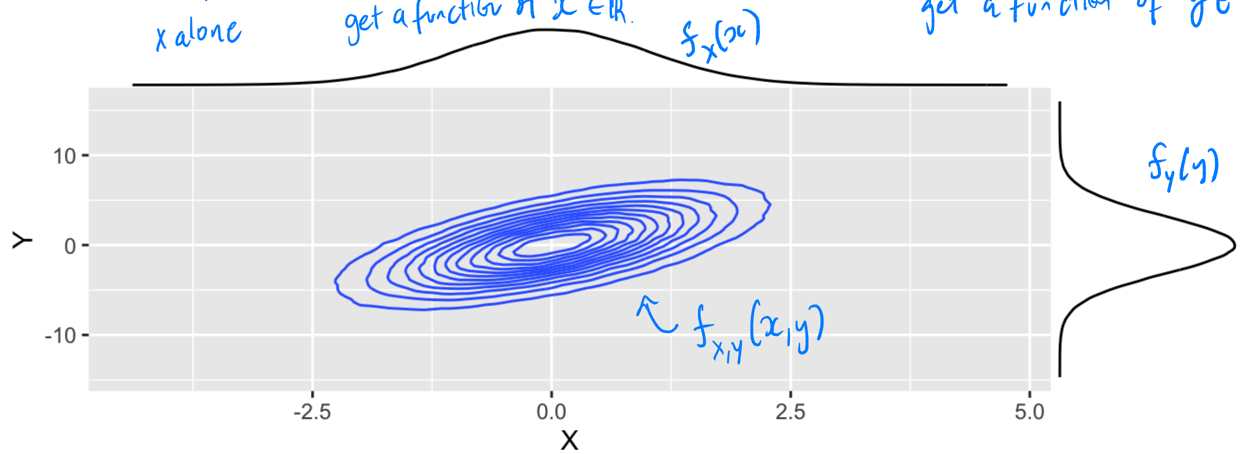
example A could be rectangle $x \in [a,b]$, $y \in [c,d]$

Example 1.9

The *marginal densities* of X and Y are given by

$$f_X(x) = \int_{\infty}^{\infty} f_{X,Y}(x,y)dy \qquad \text{and} \qquad f_Y(y) = \int_{\infty}^{\infty} f_{X,Y}(x,y)dx;$$

x alone   get a function of $x \in \mathbb{R}$.   get a function of $y \in \mathbb{R}$.



Example 1.10 (From Devore (2008) Example 5.3, pg. 187) A bank operates both a drive-up facility and a walk-up window. On a randomly selected day, let X be the proportion of time that the drive-up facility is in use and Y is the proportion of time that the walk-up window is in use.

The the set of possible values for (X, Y) (random vector) is the square $D = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$. Suppose the joint pdf is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & x \in [0,1], y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

support $\{(x,y) : 0 \le x \le 1, 0 \le y \le 1,$ not both $x, y = 0\}$

Evaluate the probability that both the drive-up and the walk-up windows are used a quarter of the time or less.

$$P(0 \le X \le \tfrac{1}{4}, 0 \le Y \le \tfrac{1}{4}) = \int_0^{1/4}\int_0^{1/4} \frac{6}{5}(x+y^2)\,dx\,dy$$

$$= \int_0^{1/4} \frac{6}{5}\left[\frac{x^2}{2} + xy^2\right]_{x=0}^{1/4} dy$$

$$= \int_0^{1/4} \frac{6}{5}\left(\frac{1}{32} + \frac{y^2}{4}\right) dy$$

$$= \left[\frac{6}{5}\left(\frac{y}{32} + \frac{y^3}{12}\right)\right]_0^{1/4} = \frac{6}{5}\left(\frac{1}{32}\cdot\frac{1}{4} + \frac{1}{12}\left(\frac{1}{4}\right)^3\right) = \frac{7}{640}$$

$$= 0.0109$$

Find the marginal densities for X and Y.

$$f_X(x) = \int_0^1 \frac{6}{5}(x+y^2)\,dy = \frac{6}{5}\left[xy + \frac{y^3}{3}\right]_{y=0}^{1} = \begin{cases} \frac{6}{5}\left(x+\frac{1}{3}\right) & \text{for } x \in [0,1] \\ 0 & \text{o.w.} \end{cases}$$

$f_Y(y)$

Leave up to you.

Compute the probability that the drive-up facility is used a quarter of the time or less.

$$P\left(X \le \frac{1}{4}\right) = \int_0^{1/4} f_X(x)\,dx = \int_0^{1/4} \frac{6}{5}\left(x+\frac{1}{3}\right)dx$$

$$= \frac{6}{5}\left[\frac{x^2}{2} + \frac{x}{3}\right]_0^{1/4}$$

$$= \frac{11}{80} = 0.1375$$

2 Expected Value and Variance

Definition 2.1 The *expected value* (average or mean) of a random variable X with pdf or pmf f_X is defined as

$$E[X] = \begin{cases} \sum\limits_{x \in \mathcal{X}} x f_X(x_i) & X \text{ is discrete} \\ \int\limits_{x \in \mathcal{X}} x f_X(x) dx & X \text{ is continuous.} \end{cases}$$

Where $\mathcal{X} = \{x : f_X(x) > 0\}$ is the support of X.

This is a weighted average of all possible values $\mathcal{X}$ by the probability distribution.

Example 2.1 Let $X \sim \text{Bernoulli}(p)$. Find $E[X]$.

X is the r.v., pmf is $f(x)$, support is $\{0,1\}$,

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{o.w.} \end{cases} \Rightarrow f(x) = \begin{cases} p & \text{when } x=1 \\ 1-p & \text{when } x=0 \end{cases} \text{ or } f(x) = p^x(1-p)^{1-x} \text{ for } x \in \{0,1\}$$

parameter is p.

$$EX = \sum_{x \in \{0,1\}} x f(x) = 0(1-p) + 1(p) = p.$$

Example 2.2 Let $X \sim \text{Exp}(\lambda)$. Find $E[X]$. cts

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

integration by parts

HW 1

$$EX = \int_0^\infty x \lambda e^{-\lambda x} dx = \dots = \frac{1}{\lambda}$$

Definition 2.2 Let $g(X)$ be a function of a continuous random variable X with pdf f_X. Then,

$$E[g(X)] = \int_{x \in \mathcal{X}} g(x) f_X(x) dx.$$

sometimes this is hard to compute analytically

$\Rightarrow$ we will need stat computing to estimate (Ch. 5)

Definition 2.3 The *variance* (a measure of spread) is defined as

$$\begin{aligned} Var[X] &= E\left[(X - E[X])^2\right] \\ &= E[X^2] - (E[X])^2 \end{aligned}$$

computational formula

$g(x) = x^2$

Example 2.3 Let X be the number of cylinders in a car engine. The following is the pmf function for the size of car engines.

x	4.0	6.0	8.0
f	0.5	0.3	0.2

i.e. $P(X=4) = 0.5$, etc.

Who might care about this?

Find

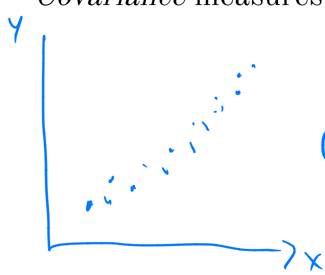
$E[X] = \sum_{x\in\mathcal{X}} x f(x) = 4\cdot 0.5 + 6\cdot 0.3 + 8\cdot 0.2 = 5.4$

$Var[X] = E(X^2) - (EX)^2$

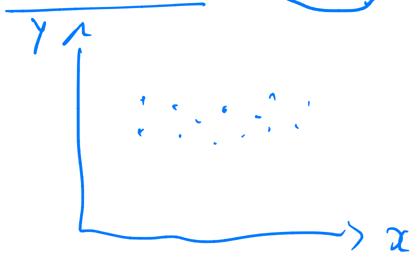
$E(X^2) = \sum_{x\in\mathcal{X}} x^2 f(x) = 4^2(0.5) + 6^2(0.3) + 8^2(0.2) = 31.6$

$\Rightarrow Var\, X = 31.6 - (5.4)^2 = 2.44$ easier to interpret is $sd(X) = \sqrt{Var(X)} = 1.56$

Covariance measures how two random variables vary together (their linear relationship).

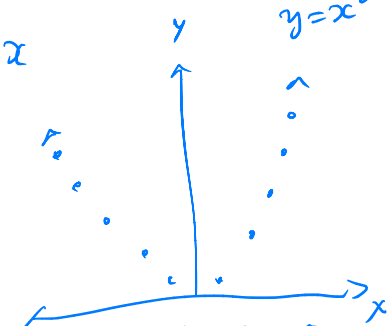


$Cov(X,Y) > 0$



"random noise"

$cov[X,Y] \approx 0$



$cov(x,y) = 0$.

Definition 2.4 The *covariance* of X and Y is defined by

$$\begin{aligned} Cov[X,Y] &= E\left[(X - E[X])(Y - E[Y])\right] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

↑ computational form

and the *correlation* of X and Y is defined as

$$\rho(X,Y) = \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}}.$$

$\rho \in [-1, 1]$

Two variables X and Y are *uncorrelated* if $\rho(X,Y) = 0$.

3 Independence and Conditional Probability

In classical probability, the *conditional probability* of an event A given that event B has occured is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Definition 3.1 Two events A and B are *independent* if $P(A|B) = P(A)$. The converse is also true, so

$$A \text{ and } B \text{ are independent} \Leftrightarrow P(A|B) = P(A) \Leftrightarrow P(A \cap B) =$$

Theorem 3.1 (Bayes' Theorem) Let A and B be events. Then,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} =$$

3.1 Random variables

The same ideas hold for random variables. If X and Y have joint pdf $f_{X,Y}(x,y)$, then the conditional density of X given $Y = y$ is

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

Thus, two random variables X and Y are independent if and only if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$

Also, if X and Y are independent, then

$$f_{X|Y=y}(x) =$$

4 Properties of Expected Value and Variance

Suppose that X and Y are random variables, and a and b are constants. Then the following hold:

1. $E[aX + b] =$

2. $E[X + Y] =$

3. If X and Y are independent, then $E[XY] =$

4. $Var[b] =$

5. $Var[aX + b] =$

6. If X and Y are independent, $Var[X + Y] =$

5 Random Samples

Definition 5.1 Random variables $\{X_1, \ldots, X_n\}$ are defined as a *random sample* from f_X if $X_1, \ldots, X_n \overset{iid}{\sim} f_X$.

Example 5.1

Theorem 5.1 If $X_1, \ldots, X_n \overset{iid}{\sim} f_X$, then

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_X(x_i).$$

Example 5.2 Let $X_1, \ldots, X_n$ be iid. Derive the expected value and variance of the sample mean $\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

6 `R` Tips

From here on in the course we will be dealing with a lot of **randomness**. In other words, running our code will return a **random** result.

> But what about reproducibility??

When we generate "random" numbers in `R`, we are actually generating numbers that *look* random, but are *pseudo-random* (not really random). The vast majority of computer languages operate this way.

This means all is not lost for reproducibility!

```r
set.seed(400)
```

Before running our code, we can fix the starting point (`seed`) of the pseudorandom number generator so that we can reproduce results.

Speaking of generating numbers, we can generate numbers (also evaluate densities, distribution functions, and quantile functions) from named distributions in `R`.

```r
rnorm(100)
dnorm(x)
pnorm(x)
qnorm(y)
```