Chapter 2: Probability for Statistical Computing

We will **briefly** review some definitions and concepts in probability and statistics that will be helpful for the remainder of the class.

Just like we reviewed computational tools (`R` and packages), we will now do the same for probability and statistics.

Note: This is not meant to be comprehensive. I am assuming you already know this and maybe have forgotten a few things.

i.e. you may need to look some things up on your own.

https://xkcd.com/892/

Alternative text: "Hell, my eighth grade science class managed to conclusively reject it just based on a classroom experiment. It's pretty sad to hear about million-dollar research teams who can't even manage that."

1 Random Variables and Probability

Definition 1.1 A *random variable* is a function that maps sets of all possible outcomes of an experiment (sample space Ω) to $\mathbb{R}$.

" real numbers
$(-\infty, \infty)$

Example 1.1

experiment Toss 2 dice.

$\Omega = \{(i,j) : i = 1,..,6;\ j = 1,..,6\}$

r.v. X = sum of the dice.

Example 1.2

experiment: Randomly select 25 deer & test for CWD

r.v. X_i {0 or 1} observe $X_1, ..., X_{25}$ $\Omega = \{+, -\ \text{CWD}\}$

r.v. $p = \sum_{i=1}^{25} X_i / 25$ is also a r.v.!

Example 1.3

experiment: Deck of cards, draw one card

r.v. X : 1 if clubs, 0 otherwise.

Ω = { values of all 52 cards in a deck }
= {AC, 2C, 3C, ..., KC,
AS, 2S, ..., KS,
AD, 2D, ..., KD,
AH, 2H, ..., KH}

Ex. 1.4
Today's high temp $= X_i$

Types of random variables –

Discrete take values in a countable set.

Ex. 1.1
X_i from Ex 1.2, X from X 1.3

Continuous take values in an uncountable set (like $\mathbb{R}$)

Ex 1.4 $X_i \in \mathbb{R}$

p from ~~Ex~~ 1.2, $p \in [0, 1]$

1.1 Distribution and Density Functions

Definition 1.2 The *probability mass function (pmf)* of a random variable X is f_X defined by

Notation: sometimes when the r.v. is obvious I will omit the subscript and write $f(x)$.

$$f_X(x) = P(X = x) \qquad \text{for any } x \in \mathbb{R}$$

where $P(\cdot)$ denotes the probability of its argument.

There are a few requirements of a **valid** pmf

1. $f(x) \geq 0$ for all $x \in \mathbb{R}$.
2. $\sum_x f(x) = 1$
3. We call $\mathcal{X} = \{x : f(x) > 0\}$ the "support" of X.

Not a requirement

Example 1.4 Let Ω = all possible values of a roll of a single die = $\{1, \ldots, 6\}$ and X be the outcome of a single roll of one die $\in \{1, \ldots, 6\}$.

Fair die

$f(1) = P(X=1) = \frac{1}{6}$

$\vdots$

$f(6) = \frac{1}{6}$

$\sum_{x \in \mathcal{X}} f(x) = \sum_{x=1}^{6} \frac{1}{6} = 1$ ✓ valid pmf.

A pmf is defined for **discrete variables**, but what about **continuous**? Continuous variables do not have positive probability mass at any single point.

Definition 1.3 The *probability density function (pdf)* of a random variable X is f_X defined by

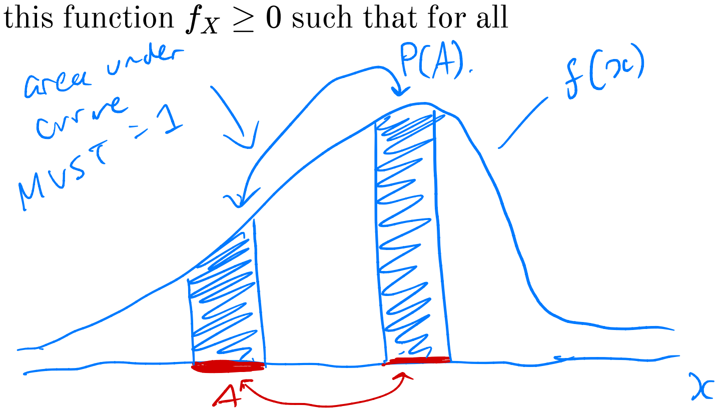
$A \subset \mathbb{R}$

$$P(X \in A) = \int_{x \in A} f_X(x)dx.$$

X is a continuous random variable if there exists this function $f_X \geq 0$ such that for all $x \in \mathbb{R}$, this probability exists.

For f_X to be a valid pdf,

1. $f(x) \geq 0 \quad \forall x \in \mathbb{R}$
2. $\int_{\mathbb{R}} f(x)dx = 1$



Again $\mathcal{X} = \{x : f(x) > 0\}$ is the "support" of X

There are many named pdfs and ~~pdfs~~ pmfs that you have seen in other class, e.g.

Bernoulli (pmf), Poisson (pmf), Gamma (pdf), Normal (pdf), Beta, exponential, hypergeometric.

Example 1.5 Let

$$f(x) = \begin{cases} c(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find c and then find $P(X > 1)$

↳ such that $f(x)$ is a valid pdf.

$$1 = \int_0^2 c(4x - 2x^2)\,dx = c\left[2x^2 - \frac{2x^3}{3}\right]_0^2 = c\left[\frac{8}{3}\right] \Rightarrow c = \frac{3}{8}$$

"normalizing constant"

$$P(X > 1) = \int_1^\infty f(x)\,dx = \int_1^2 \frac{3}{8}(4x - 2x^2)\,dx = \frac{3}{8}\left[2x^2 - \frac{2x^3}{3}\right]_1^2 = \frac{1}{2}$$

Definition 1.4 The *cumulative distribution function (cdf)* for a random variable X is F_X defined by

$$F_X(x) = P(X \leq x), \quad x \in \mathbb{R}.$$

The cdf has the following properties

1.
2.
3.

A random variable X is *continuous* if F_X is a continuous function and *discrete* if F_X is a step function.

Example 1.6 Find the cdf for the previous example.

Note $f(x) = F'(x) = \frac{dF(x)}{dx}$ in the continuous case.

1.2 Two Continuous Random Variables

Definition 1.5 The *joint pdf* of the continuous vector (X, Y) is defined as

$$P((X,Y) \in A) = \iint_A f_{X,Y}(x,y)dxdy$$

for any set $X \subset \mathbb{R}^2$.

Joint pdfs have the following properties

1.

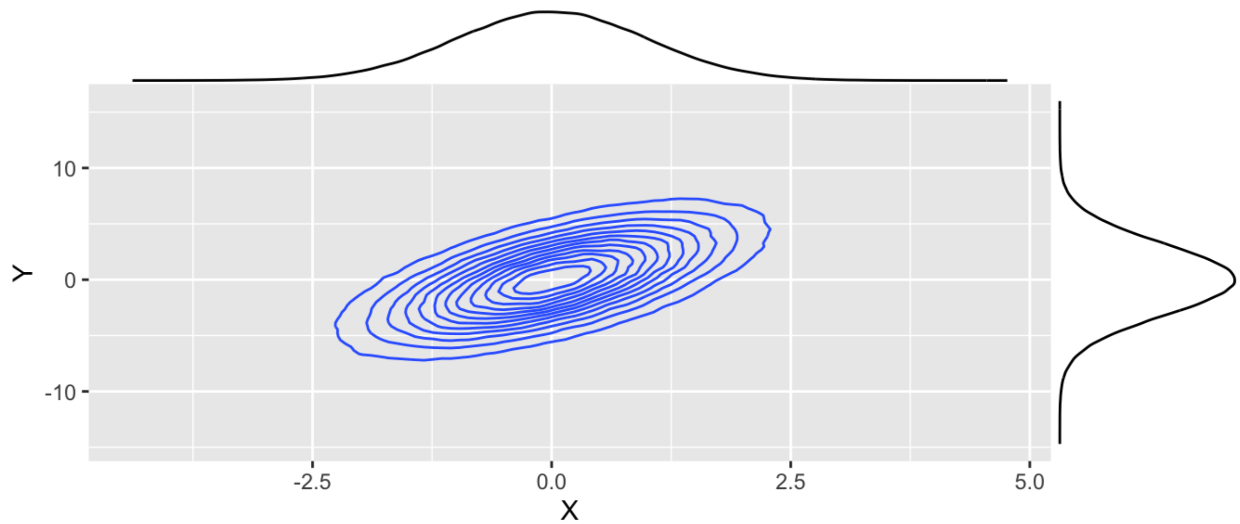
2.

and a support defined to be $\{(x,y) : f_{X,Y}(x,y) > 0\}$.

Example 1.7

The *marginal densities* of X and Y are given by

$$f_X(x) = \int_{\infty}^{\infty} f_{X,Y}(x,y)dy \qquad \text{and} \qquad f_Y(y) = \int_{\infty}^{\infty} f_{X,Y}(x,y)dx;$$



Example 1.8 (From Devore (2008) Example 5.3, pg. 187) A bank operates both a drive-up facility and a walk-up window. On a ramdonly selected day, let X be the proportion of time that the drive-up facility is in use and Y is the proportion of time that the walk-up window is in use.

The the set of possible values for (X, Y) is the square $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Suppose the joint pdf is given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{6}{5}(x^2 + y^2) & x \in [0, 1], y \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Evaluate the probability that both the drive-up and the walk-up windows are used a quarter of the time or less.

Find the marginal densities for X and Y.

Compute the probability that the drive-up facility is used a quarter of the time or less.

2 Expected Value and Variance

Definition 2.1 The *expected value* (average or mean) of a random variable X with pdf or pmf f_X is defined as

$$E[X] = \begin{cases} \sum\limits_{x \in \mathcal{X}} x f_X(x_i) & X \text{ is discrete} \\ \int\limits_{x \in \mathcal{X}} x f_X(x) dx & X \text{ is continuous.} \end{cases}$$

Where $\mathcal{X} = \{x : f_X(x) > 0\}$ is the support of X.

This is a weighted average of all possible values $\mathcal{X}$ by the probability distribution.

Example 2.1 Let $X \sim \text{Bernoulli}(p)$. Find $E[X]$.

Example 2.2 Let $X \sim \text{Exp}(\lambda)$. Find $E[X]$.

Definition 2.2 Let $g(X)$ be a function of a continuous random variable X with pdf f_X. Then,

$$E[g(X)] = \int_{x \in \mathcal{X}} g(x) f_X(x) dx.$$

Definition 2.3 The *variance* (a measure of spread) is defined as

$$\begin{aligned} Var[X] &= E\left[(X - E[X])^2\right] \\ &= E[X^2] - (E[X])^2 \end{aligned}$$

Example 2.3 Let X be the number of cylinders in a car engine. The following is the pmf function for the size of car engines.

x	4.0	6.0	8.0
f	0.5	0.3	0.2

Find

$E[X]$

$Var[X]$

Covariance measures how two random variables vary together (their linear relationship).

Definition 2.4 The *covariance* of X and Y is defined by

$$\begin{aligned} Cov[X, Y] &= E\left[(X - E[X])(Y - E[Y])\right] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

and the *correlation* of X and Y is defined as

$$\rho(X,Y) = \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}}.$$

Two variables X and Y are *uncorrelated* if $\rho(X,Y) = 0$.

3 Independence and Conditional Probability

In classical probability, the *conditional probability* of an event A given that event B has occured is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Definition 3.1 Two events A and B are *independent* if $P(A|B) = P(A)$. The converse is also true, so

$$A \text{ and } B \text{ are independent} \Leftrightarrow P(A|B) = P(A) \Leftrightarrow P(A \cap B) = .$$

Theorem 3.1 (Bayes' Theorem) Let A and B be events. Then,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = .$$

3.1 Random variables

The same ideas hold for random variables. If X and Y have joint pdf $f_{X,Y}(x, y)$, then the conditional density of X given $Y = y$ is

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x, y)}{f_Y(y)}.$$

Thus, two random variables X and Y are independent if and only if

$$f_{X,Y}(x, y) = f_X(x) f_Y(y).$$

Also, if X and Y are independent, then

$$f_{X|Y=y}(x) =$$

4 Properties of Expected Value and Variance

Suppose that X and Y are random variables, and a and b are constants. Then the following hold:

1. $E[aX + b] =$

2. $E[X + Y] =$

3. If X and Y are independent, then $E[XY] =$

4. $Var[b] =$

5. $Var[aX + b] =$

6. If X and Y are independent, $Var[X + Y] =$

5 Random Samples

Definition 5.1 Random variables $\{X_1, \dots, X_n\}$ are defined as a *random sample* from f_X if $X_1, \dots, X_n \overset{iid}{\sim} f_X$.

Example 5.1

Theorem 5.1 If $X_1, \dots, X_n \overset{iid}{\sim} f_X$, then

$$f(x_1, \dots, x_n) = \prod_{i=1}^{n} f_X(x_i).$$

Example 5.2 Let $X_1, \dots, X_n$ be iid. Derive the expected value and variance of the sample mean \$$n =$ {i = 1}^n X_i.

6 `R` Tips

From here on in the course we will be dealing with a lot of **randomness**. In other words, running our code will return a **random** result.

> But what about reproducibility??

When we generate "random" numbers in `R`, we are actually generating numbers that *look* random, but are *pseudo-random* (not really random). The vast majority of computer languages operate this way.

This means all is not lost for reproducibility!

```
set.seed(400)
```

Before running our code, we can fix the starting point (`seed`) of the pseudorandom number generator so that we can reproduce results.

Speaking of generating numbers, we can generate numbers (also evaluate densities, distribution functions, and quantile functions) from named distributions in `R`.

```
rnorm(100)
dnorm(x)
pnorm(x)
qnorm(y)
```

7 Food for thought

Recall an indicator function is defined as

$$1_{\{A\}} = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{otherwise} \end{cases}.$$

Example 7.1

Example 7.2 If $X \sim N(0,1)$, the pdf is $f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ for $-\infty < x < \infty$.

If $f(x) = \frac{c}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) 1_{\{x>0\}}$, what is c?